Master’s thesis

Almost irreducible subalgebras of simple Lie algebras

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Abstract

The aim of this master’s thesis is to determine the irreducible representations of a simple Lie algebra over an algebraically closed field of characteristic zero that decompose in at most two irreducible summands, when restricted to a simple subalgebra. We will study the case of a Lie algebra of type $G_2$, whose long roots form a root system of type $A_2$, and the embedding of a Lie algebra of type $B_n$ into a Lie algebra of type $D_{n+1}$. For this purpose, we will recall some basic notions about Lie algebras and their representation theory, and give some specific results for our work. We will especially explain an inductive method to determine these irreducible representations, for the case of $B_n$ in $D_{n+1}$.

Acknowledgement

I would like to thank Professor Donna Testerman for her support and for the long time she spend helping me through the realisation of this project. I am grateful that she suggests me this great subject for my master’s thesis, that I really enjoyed. I would also like to thank Mikaël Cavallin for his useful advices and for his help.
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Introduction

In this project, we are interested in studying the restriction of an irreducible representation of a simple Lie algebra to a simple subalgebra. In 1952, E.B. Dynkin determined all the cases in which an irreducible representation of a semisimple Lie algebra stays irreducible upon restriction to a semisimple subalgebra, in his article [Dyn].

On the other hand, K. Koike and I. Terada published in 1985 an article (see [Koi]) giving a combinatorial method to enumerate the number of irreducible summands appearing in the restriction of an irreducible representation, for the simple Lie algebras of type $B_n$, $C_n$, and $D_n$.

Here, the aim will be to determine when a representation of a simple Lie algebra decomposes into at most two irreducible summands, when restricted to a simple subalgebra. We will study two chosen cases, namely the embedding of $A_2$ into $G_2$, and the embedding of $B_n$ into $D_{n+1}$ for $n \geq 2$.

For this purpose, we first introduce the basic definitions and results about representation theory of semisimple Lie algebras, and some useful formulas for the computation of the multiplicities of weights or the dimensions of representations. This will then be enough to work on the case of $A_2$ embedded into $G_2$, with the embedding given by the long roots of $G_2$ forming a root system of type $A_2$.

Then we will explain the inductive method that we will use for the study of the next case. In this part, we will need to study some parabolic and Levi subalgebras, in order to use the results obtained for smaller rank Lie algebras.

We will finally study the cases of $B_2$ in $D_3$ and $B_3$ in $D_4$, before moving to the inductive step. Here, we will be able to determine which representations of $D_{n+1}$ decompose into at most two irreducible summands when restricted to $B_n$, using the case of $B_{n-1}$ into $D_n$. 
Chapter 1

Results on semisimple Lie algebras

Throughout this project, we will work with finite-dimensional semisimple Lie algebras over an algebraically closed field $K$ of characteristic zero. In this chapter, we start by a few general results about semisimple Lie algebras and their representation theory.

We will mention some results without proving them, but all the details can be found in chapters I, II, III and VI from [Hum] and in paragraph 6 from [Bou1].

1.1 Structure of semisimple Lie algebras

We recall first some basic definitions. The notation $L$ will always denote a Lie algebra over $K$, with $[-,-]$ the Lie bracket.

**Definition 1.1.1.** A Lie algebra $L$ is simple if $[L,L] \neq 0$ and if $L$ has no ideals except $\{0\}$ and $L$.

**Definition 1.1.2.** Let $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ denote the derived series of $L$. We say that $L$ is solvable if there exists $n$ such that $L^{(n)} = \{0\}$.

**Definition 1.1.3.** The radical $\text{rad}(L)$ of $L$ is its unique maximal solvable ideal.

**Definition 1.1.4.** A Lie algebra $L$ is semisimple if $\text{rad}(L) = \{0\}$.

We recall that a semisimple Lie algebra $L$ can always be identified with a linear Lie algebra, i.e., a subalgebra of $\text{End}(V)$ for a $K$-vector space $V$, with the bracket given by $[x,y] = xy -yx$ for every $x,y \in \text{End}(V)$.

**Definition 1.1.5.** Let $x$ be an element of a semisimple Lie algebra $L$. We say that $x$ is a nilpotent element if $x$ is a nilpotent endomorphism, when looking at $x$ as an element of $\text{End}(V)$.

**Remark.** One can show that the notion of a nilpotent element is well-defined, that is, the definition does not depend on the choice of the $K$-vector space $V$.

**Definition 1.1.6.** Let $x \in L$, we define $\text{ad}_x(y) = [x,y]$ for every $y \in L$, so $\text{ad}_x$ is the adjoint representation applied to $x$, i.e., $\text{ad}_x : L \to L$ is a linear map.

**Lemma 1.1.7.** Let $x$ be an element of a semisimple Lie algebra $L$. Then $x$ is a nilpotent element of $L$ if and only if $\text{ad}_x$ is a nilpotent endomorphism of $L$.

**Proof.** See corollary in section 3, paragraph 6 from [Bou1].
1.1.1 Root systems

Let $L$ be a semisimple Lie algebra.

**Definition 1.1.8.** Let $x \in L$. We say that $x$ is semisimple if $\text{ad}x$ is semisimple, i.e., if $\text{ad}x$ is diagonalizable.

**Definition 1.1.9.** A subalgebra $H$ of $L$ is a toral subalgebra if $H$ consists of semisimple elements.

**Proposition 1.1.10.** Let $h$ be a maximal toral subalgebra of $L$. Then we can decompose $L$ as $L = h + \bigoplus_{\alpha \in \Phi} L_{\alpha}$, where $\alpha \in h^*$, $L_{\alpha} = \{x \in L | [h,x] = \alpha(h)x \forall h \in h\}$ and $\Phi = \{\alpha \in h^* | L_{\alpha} \neq \{0\}, \alpha \neq 0\}$, with $h^*$ denoting the dual space of $h$.

**Proof.** See section 8 from [Hum].

**Definition 1.1.11.** The elements $\alpha \in \Phi$ such that $L_{\alpha} \neq \{0\}$ are called roots of $L$ relative to $h$.

**Lemma 1.1.12.** For $\alpha, \beta \in \Phi$, we have $[L_{\alpha}, L_{\beta}] \subset L_{\alpha + \beta}$.

**Proof.** It is a consequence of the definition of $L_{\alpha}$ for $\alpha \in \Phi$ and from the Jacobi identity.

**Definition 1.1.13.** Let $E$ be a euclidean space with scalar product $(\cdot, \cdot)$. We define a product $<\alpha, \beta> = \frac{2(\alpha, \beta)}{(\beta, \beta)}$, which is bilinear in the first variable. Then we set $\sigma_{\alpha}(\beta) = \beta - <\beta, \alpha> \alpha$, the reflection through $\alpha$. Now we say that a subset $\Psi$ of $E$ is a root system in $E$ if

1) $\Psi$ is finite, spans $E$ and does not contain 0,

2) for $\alpha \in \Psi$, the only multiples of $\alpha \in \Psi$ are $\alpha$ and $-\alpha$,

3) for $\alpha \in \Psi$, the reflection $\sigma_{\alpha}$ is such that $\sigma_{\alpha}(\Psi) = \Psi$,

4) for $\alpha, \beta \in \Psi$, we have $<\alpha, \beta> \in \mathbb{Z}$.

In order to see that the set $\Phi$ defined in 1.1.10 is a root system with these axioms, we need to have a euclidean space and a scalar product on it.

**Definition 1.1.14.** The Killing form on $L$ is the application $\kappa : L \times L \rightarrow K$ defined by $\kappa(x,y) = \text{Trace}(\text{ad}x \text{ad}y)$.

When $L$ is a semisimple Lie algebra, the Killing form is non-degenerate on $L$. Moreover, its restriction to $h$ is still non-degenerate, which allows the next definition.

**Definition 1.1.15.** Let $\alpha \in h^*$. We set $t_{\alpha} \in h$ the element such that $\alpha(h) = \kappa(t_{\alpha}, h)$ for every $h \in h$. From that we define an element $h_{\alpha} \in h$ as $h_{\alpha} = \frac{2t_{\alpha}}{\kappa(t_{\alpha}, t_{\alpha})}$.

**Definition 1.1.16.** Let $\alpha, \beta \in h^*$. We define a non-degenerate bilinear form on $h^*$ by $(\alpha \beta) = \kappa(t_{\alpha}, t_{\beta})$. 
Let now $E_{\mathbb{R}}$ be the $\mathbb{R}$-space spanned by all the elements of $\Phi$. Then the set $\Phi$ as defined in 1.1.10 is a root system of $E_{\mathbb{R}}$ with scalar product $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$. The details are given in section 8.5 from [Hum], especially the fact that we can construct $E_{\mathbb{R}}$ as a $\mathbb{R}$-space, even when we start from a Lie algebra over any algebraically closed field of characteristic zero.

From now, and since we have seen that the two definitions of a root systems are equivalent, we will only write $E$ for the euclidean space associated to $\Phi$. When we are interested in a generating subspace of $E$ and not of $\mathfrak{h}^*$, we can have a stronger definition of a basis.

**Definition 1.1.17.** Let $\Delta$ be a subset of $\Phi$. Then $\Delta$ is a base of $\Phi$ if $\Delta$ is a basis of $E$ and if for every $\beta \in \Phi$, we can express $\beta$ as a $\mathbb{Z}$-linear combination of the elements of $\Delta$, with all coefficients nonnegative or all nonpositive.

The proof that every root system has a base can be found in section 10.1 of [Hum].

**Definition 1.1.18.** The elements of a base $\Delta$ of $\Phi$ are called the simple roots of $\Phi$.

**Definition 1.1.19.** A root $\beta$ is a positive root of $\Phi$ if we can express it as a linear combination of the simple roots with nonnegative coefficients, and similarly, $\beta$ is a negative root if it is a linear combination of the simple roots with nonpositive coefficients. We denote by $\Phi^+$ the set of positive roots of $\Phi$.

**Definition 1.1.20.** We define a partial ordering $<$ on the elements of $E$ as $\alpha < \beta$ if and only if $\beta - \alpha$ is a sum of positive roots or $\alpha = \beta$.

In the axiomatic definition of a root system, we defined the reflection through $\alpha$ as $\sigma_\alpha(\beta) = \beta - <\beta, \alpha> \alpha$, for $<\alpha, \beta> = \frac{2(\alpha, \beta)}{(\beta, \beta)}$. From these reflections we define the Weyl group associated to $\Phi$.

**Definition 1.1.21.** The subgroup of $GL(E)$ generated by the reflections $\sigma_\alpha$ for $\alpha \in \Phi$ is called the Weyl group of $\Phi$, and denoted by $W$.

**Lemma 1.1.22.** The Weyl group $W$ is a finite group, stabilizes $\Phi$ and is generated by the reflections $\sigma_\alpha$ for $\alpha \in \Delta$.

**Proof.** See chapter 10 from [Hum].

**Definition 1.1.23.** Let $\alpha$ be a root. We say that $\alpha$ has length $(\alpha, \alpha)^{1/2}$, for $(-,-)$ the scalar product on $E$.

In order to make a further step in the direction of the classification of the simple Lie algebras, we will define irreducible root systems, and it will then be enough to classify only these root systems, since every other system can be expressed as the disjoint union of irreducible ones.

**Definition 1.1.24.** Let $\Phi$ be a root system. We say that $\Phi$ is irreducible if we cannot write $\Phi$ as the union $\Phi_1 \cup \Phi_2$ of two nonempty subsets such that $\langle \alpha_1, \alpha_2 \rangle = 0$ for every $\alpha_1 \in \Phi_1, \alpha_2 \in \Phi_2$.

Equivalently, we say that $\Phi$ is irreducible if its base $\Delta$ can not be expressed as the union of two nonempty set $\Delta_1$ and $\Delta_2$ with $\langle \alpha_1, \alpha_2 \rangle = 0$ for every $\alpha_1 \in \Delta_1, \alpha_2 \in \Delta_2$. 


Proposition 1.1.25. Let $\Phi$ be a root system, with associated euclidean space $E$. Then $\Phi$ decomposes uniquely as the union of irreducible root systems $\Phi_1, \ldots, \Phi_n$ such that $E = E_1 \oplus \cdots \oplus E_n$, where $E_i$ is the subspace of $E$ associated to $\Phi_i$.

Proof. See section 11.3 from [Hum].

Lemma 1.1.26. Let $\Phi$ be an irreducible root system. Then there are at most two distinct root lengths in $\Phi$.

Proof. See section 10.4 from [Hum].

This lemma justifies the next definition:

Definition 1.1.27. Let $\Phi$ be an irreducible root system. Then we say that the roots with the maximal length are the long roots of $\Phi$, and the others (if they exist) are the short roots of $\Phi$.

1.1.2 Cartan and Borel subalgebras

Definition 1.1.28. Let $H$ be a subalgebra of $L$. We say that $H$ is a Cartan subalgebra of $L$ if $H$ is nilpotent and if the normalizer $N_L(H)$ of $H$ in $L$ is exactly $H$.

We will see that the decomposition of $L$ that we made in 1.1.10 with respect to a maximal toral subalgebra of $L$ can also be carried out with respect to a Cartan subalgebra, thanks to the next theorem. We point out that this theorem holds only in characteristic zero.

Theorem 1.1.29. Let $L$ be a semisimple Lie algebra. Then the Cartan subalgebras of $L$ are exactly the maximal toral subalgebras of $L$.

Proof. See section 15.3 from [Hum].

From now, we will denote the Cartan subalgebra of $L$ by $\mathfrak{h}$. The choice of the Cartan subalgebra for the decomposition made in 1.1.10 does not matter, as we see in the next result:

Theorem 1.1.30. Let $\mathfrak{h}_1$, $\mathfrak{h}_2$ be two Cartan subalgebras of $L$. Then there exists an automorphism $f : L \to L$ such that $f(\mathfrak{h}_1) = \mathfrak{h}_2$.

Proof. See section 16.4 from [Hum].

In fact, for our work, we will need to introduce a new family of subalgebras of $L$.

Definition 1.1.31. A maximal solvable subalgebra $\mathfrak{b}$ of $L$ is called a Borel subalgebra of $L$.

When $L$ is semisimple, we fix a Cartan subalgebra $\mathfrak{h}$ of $L$ and a base $\Delta$ of the associated root system $\Phi$, then we know that we can decompose $L$ as $L = \mathfrak{h} \oplus \oplus_{\alpha \in \Phi} L_\alpha$. Set $\mathfrak{b} = \mathfrak{h} + \oplus_{\alpha > 0} L_\alpha$. Then $\mathfrak{b}$ is a Borel subalgebra, as explained in section 16.3 from [Hum], and we call such a $\mathfrak{b}$ the standard Borel subalgebra of $L$ relative to $\mathfrak{h}$ and to the choice of the base $\Delta$.

We have a result analogous to the theorem 1.1.30 for the Borel subalgebras.

Theorem 1.1.32. Let $\mathfrak{b}_1$, $\mathfrak{b}_2$ be two Borel subalgebras of $L$. Then there exists an automorphism $f : L \to L$ such that $f(\mathfrak{b}_1) = \mathfrak{b}_2$.

Proof. See section 16.4 from [Hum].
1.1.3 Classification of the simple Lie algebras

In this section, we will state the classification of simple Lie algebras, and so \( L \) will here denote a simple Lie algebra.

**Definition 1.1.33.** Let \( \Phi \) be a root system. We fix an ordering \( \alpha_1, \ldots, \alpha_n \) of the simple roots and define the Cartan matrix of \( \Phi \) as the matrix with entry \( \langle \alpha_i, \alpha_j \rangle \) in row \( i \) and column \( j \).

**Remark.** We also say that a matrix \( A \) is the Cartan matrix of \( L \) when \( A \) is the Cartan matrix of a root system \( \Phi \) of \( L \).

**Definition 1.1.34.** Let \( \Phi \) be a root system. We fix an ordering \( \alpha_1, \ldots, \alpha_n \) of the simple roots, as for the Cartan matrix. Then the Dynkin diagram of \( \Phi \) is a graph with \( n \) vertices, and the vertex \( i \) and \( j \) joined by \( \langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle \) edges for \( i \neq j \). Moreover, if there is more than one edge between the vertices \( i \) and \( j \), we add an arrow pointing towards the index of the shorter of the two roots.

We can check (see section 9.4 from [Hum]) that \( \langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle \) is equal to 0, 1, 2 or 3 for \( i \neq j \). And it is direct to see that a root system is irreducible if and only if the corresponding Dynkin diagram is connected.

**Theorem 1.1.35.** (Classification of the irreducible root systems). Let \( \Phi \) be an irreducible root system. Then its Dynkin diagram is of one of the following type, and there exists an irreducible root system for each of them:

**Proof.** See section 11.4 from [Hum].

**Remark.** The restriction on \( n \) for types \( A_n, B_n, C_n \) and \( D_n \) are here to get the unicity of each Dynkin diagram, since a type \( B_2 \) or \( C_2 \) would be the same, or \( A_3 \) and \( D_3 \) for example. However, it is sometimes convenient to consider \( A_3 \) as being of type \( D_3 \), for example.

**Definition 1.1.36.** Let \( \Phi \) and \( \Psi \) be two root systems, spanning euclidean spaces \( E \) and \( F \), respectively, and let \( \Delta \) be a base of \( \Phi \). We say that \( \Phi \) and \( \Psi \) are isomorphic as root systems if there is a bijective linear map \( f : E \rightarrow F \) such that for every \( \alpha, \beta \in \Delta \), we have \( \langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle \).

We can show that for every root system \( \Phi \), there exists a semisimple Lie algebra whose root system is exactly \( \Phi \). And if \( \Phi \) is irreducible, the Lie algebra is simple. Moreover, if \( L \) and \( L' \) are semisimple Lie algebras with root systems \( \Phi \) and \( \Phi' \) and if \( \Phi \) and \( \Phi' \) are isomorphic as root systems, then \( L \) and \( L' \) are also isomorphic. So from the classification of irreducible root system, we deduce a classification of simple Lie algebras.

**Theorem 1.1.37.** (Classification of the simple Lie algebras). Let \( L \) be a simple Lie algebra. Then the root system of \( L \) has a connected Dynkin diagram of type \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 \) or \( G_2 \). If \( L \) and \( L' \) are two Lie algebras whose associated Dynkin diagram is the same, then \( L \) and \( L' \) are isomorphic.

Now, since we know that there is a one-to-one correspondence between simple Lie algebras and types of Dynkin diagram, we will speak of a Lie algebra of type \( A_n \) to denote a simple Lie algebra with root system whose corresponding Dynkin diagram is of type \( A_n \) (similarly for every type of diagram).
CHAPTER 1. RESULTS ON SEMISIMPLE LIE ALGEBRAS

Table 1.1: Classification of the irreducible root systems.

1.1.4 Universal enveloping algebra

Let $A$ be an associative unital algebra. Then we can construct a structure of Lie algebra on $A$, by defining a Lie bracket as the commutator of two elements, i.e., for $x, y \in L$ we set $[x, y] = xy -yx$.

Let $L$ be a Lie algebra. The aim of the construction of the universal enveloping algebra is to find an associative unital algebra such that the Lie bracket of $L$ corresponds with the commutator of this new algebra. We will give a characterisation by universal property of the universal enveloping algebra, but also an explicit construction, and we will see which relation links the Lie algebra to its universal enveloping algebra.

All algebras considered here are algebras over the same algebraically closed field of characteristic zero.

**Definition 1.1.38.** Let $L$ be a Lie algebra. A universal enveloping algebra of $L$ is an associative unital algebra $\mathfrak{U}(L)$ with a linear map $i : L \to \mathfrak{U}(L)$ such that $i([xy]) = i(x)i(y) - i(y)i(x)$ for every $x, y \in L$ and the following universal property holds:
For any associative unital algebra $A$ and any linear map $f : L \rightarrow A$ with $f([xy]) = f(x)f(y) - f(y)f(x)$ for every $x, y \in L$, then there exists a unique morphism of algebras $g : \mathfrak{U}(L) \rightarrow A$ such that $g \circ i = f$.

**Remark.** From this property, we deduce that the universal enveloping algebra of a given Lie algebra $L$ is unique up to isomorphism. Indeed, if we suppose that $\mathfrak{U}(L)$ and $\mathfrak{V}(L)$ are two such algebras, with associated linear maps $i$ and $j$, we apply the universal property twice, once for $\mathfrak{U}(L)$ as the enveloping algebra and $\mathfrak{V}(L)$ in place of $A$, and once exchanging $\mathfrak{U}(L)$ and $\mathfrak{V}(L)$. We have then the two commutative diagrams:

\[
\begin{array}{ccc}
L & \xrightarrow{j} & \mathfrak{V}(L) \\
\downarrow{i} & \phi \downarrow & \\
\mathfrak{U}(L) & & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
L & \xrightarrow{i} & \mathfrak{U}(L) \\
\downarrow{j} & \psi \downarrow & \\
\mathfrak{V}(L) & & \\
\end{array}
\]

so we have $\phi \circ i = j$ and $\psi \circ j = i$. Then applying the universal property to $\mathfrak{U}(L)$ with $A = \mathfrak{U}(L)$, we have

\[
\begin{array}{ccc}
L & \xrightarrow{i} & \mathfrak{U}(L) \\
\downarrow{i} & \Id_{\mathfrak{U}(L)} \downarrow & \\
\mathfrak{U}(L) & & \\
\end{array}
\]

but we also know that $(\psi \circ \phi) \circ i = \psi \circ j = i$, so by unicity of the morphism, we get $\psi \circ \phi = \Id_{\mathfrak{U}(L)}$. Similarly for $\mathfrak{V}(L)$, we obtain $\phi \circ \psi = \Id_{\mathfrak{V}(L)}$, meaning that the universal enveloping algebra is unique up to isomorphism.

Now we can give an explicit construction for the universal enveloping algebra of $L$, which also shows its existence. For $L$ a finite-dimensional Lie algebra, let $T(L)$ be its tensor algebra, i.e.,

\[
T(L) = \sum_{n=0}^{\infty} L \otimes^{n},
\]

with the convention that $L \otimes^{0}$ is just the field. Then let $J$ be the ideal of $T(L)$ generated by

\[
\{ x \otimes y - y \otimes x - [xy] \mid x, y \in L \}.
\]

From that we construct the universal enveloping algebra $\mathfrak{U}(L)$ as the quotient $T(L)/J$.

Let $i : L \rightarrow \mathfrak{U}(L)$ be defined by the restriction of the canonical surjection $\pi : T(L) \rightarrow T(L)/J$ to $L$. By the universal property of the tensor algebra $T(L)$, we can show that $\mathfrak{U}(L)$ with the linear map $i$ is the universal enveloping algebra of $L$. For more details, see section 17.2 from [Hum].

With this construction we have a map $\sigma : L \rightarrow \mathfrak{U}(L)$, given by the inclusion of $L$ in $T(L)$ followed by the quotient.

**Theorem 1.1.39. (Poincaré-Birkhoff-Witt, or PBW.)** The map $\sigma : L \rightarrow \mathfrak{U}(L)$, given by the inclusion of $L$ in $T(L)$ and the quotient by $J$, is injective.
Proof. See sections 17.3 and 17.4 from [Hum]. (The PBW theorem is stated differently in [Hum], and our result is given as a corollary.)

With this theorem, we can now see the Lie algebra as a subalgebra of its universal enveloping algebra, identifying $L$ with its image $\sigma(L)$, if we define a Lie bracket on $\mathfrak{U}(L)$ given by the commutators, as desired at the beginning of this section.

As explained in section 1, paragraph 3 from [Bou1], there is a bijective correspondence between the representations of a Lie algebra $L$ and the representations of its universal enveloping algebra $\mathfrak{U}(L)$. In particular, we can extend every representation of $L$ to a representation of $\mathfrak{U}(L)$.

1.1.5 Generators of a simple Lie algebra

Let $L$ be a semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h}$, corresponding root system $\Phi$ and simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Let $h_{\alpha_i}$ be the element of $\mathfrak{h}$ defined in 1.1.15.

**Lemma 1.1.40.** Let $\alpha_i$ be a simple root, and $e_{\alpha_i}$ be a nonzero element of $L_{\alpha_i}$. Then there exists a unique element $f_{\alpha_i} \in L_{-\alpha_i}$ such that $[e_{\alpha_i}, f_{\alpha_i}] = h_{\alpha_i}$.

**Proof.** It is clear since $h_{\alpha_i} \in [L_{\alpha_i}, L_{-\alpha_i}]$, by section 8.3 from [Hum].

**Proposition 1.1.41.** The semisimple Lie algebra $L$ is generated by
\[
\{e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} \mid 1 \leq i \leq n\}.
\]
Moreover, these generators satisfy the relations:

1) $[h_{\alpha_i}, h_{\alpha_j}] = 0$ for all $1 \leq i, j \leq n$,
2) $[e_{\alpha_i}, f_{\alpha_i}] = h_{\alpha_i}$ for all $1 \leq i \leq n$,
3) $[e_{\alpha_i}, f_{\alpha_j}] = 0$ if $i \neq j$,
4) $[h_{\alpha_i}, e_{\alpha_j}] = \langle \alpha_j, \alpha_i \rangle e_{\alpha_j}$ for all $1 \leq i, j \leq n$,
5) $[h_{\alpha_i}, f_{\alpha_j}] = -\langle \alpha_j, \alpha_i \rangle f_{\alpha_j}$ for all $1 \leq i, j \leq n$.

**Proof.** The fact that the set $\{e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} \mid 1 \leq i \leq n\}$ generates $L$ can be deduced from the decomposition of $L$ as
\[
L = \mathfrak{h} + (\bigoplus_{\alpha \in \Phi} L_\alpha).
\]
Using the fact that the simple roots $\alpha_1, \ldots, \alpha_n$ form a base of $\Phi$, the relation $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ when $\alpha, \beta, \alpha + \beta \in \Phi$ and the fact that the Cartan subalgebra $\mathfrak{h}$ is the sum $\sum_{\alpha \in \Phi} [L_\alpha, L_{-\alpha}]$, we get the result.

For the relations, see section 18.1 from [Hum].

From now, we will use the notation $e_\alpha$ (respectively $f_\alpha$) to denote an element of $L_\alpha$ (respectively $L_{-\alpha}$) such that the Lie algebra $L$ is generated by $\{e_\alpha, f_\alpha, h_\alpha \mid \alpha \in \Delta\}$, with $h_\alpha = [e_\alpha, f_\alpha]$. 
1.2 Representation theory

For this section, we fix a semisimple Lie algebra \( L \), a Cartan subalgebra \( \mathfrak{h} \) of \( L \), \( \Phi \) the associated root system with base \( \Delta \), and \( W \) the Weyl group. The notation \( \mathfrak{h}^* \) denotes the dual space of \( \mathfrak{h} \).

**Definition 1.2.1.** An \( L \)-module is completely reducible if it is a direct sum of irreducible \( L \)-modules.

**Theorem 1.2.2. (Weyl’s theorem of complete reducibility).** Let \( L \) be a semisimple Lie algebra, and \( V \) be an \( L \)-module. Then \( V \) is a completely reducible \( L \)-module.

**Proof.** See section 6.3 from [Hum]. \( \square \)

1.2.1 Abstract theory of weights

Let \( E \) be the euclidean space generated by the root system \( \Phi \), and \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) a fixed base of \( \Phi \).

**Definition 1.2.3.** Let \( \lambda \in E \).

a) If \( \lambda \) is such that \( \langle \lambda, \alpha \rangle \in \mathbb{Z} \) for every \( \alpha \in \Phi \), we say that \( \lambda \) is a weight. We will denote by \( \Lambda \) the set of weights in \( E \).

b) For \( \lambda \) a weight, we say that \( \lambda \) is dominant if \( \langle \lambda, \alpha_i \rangle \geq 0 \) for every \( \alpha_i \in \Delta \).

c) The elements \( \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \) with \( 1 \leq i \leq n \) also form a basis of \( E \). Let \( \lambda_1, \ldots, \lambda_n \) be the dual basis of \( E \) with respect to this basis and relative to the scalar product \( \langle \cdot, \cdot \rangle \). Then the \( \lambda_i \) are dominant weights, and we call them the fundamental dominant weights corresponding to \( \alpha_1, \ldots, \alpha_n \).

In fact, for showing that \( \lambda \) is a weight, it is enough to check that \( \langle \lambda, \alpha_i \rangle \in \mathbb{Z} \) for every \( \alpha_i \in \Delta \), since every root is a \( \mathbb{Z} \)-linear combination of the elements of \( \Delta \).

**Lemma 1.2.4.** With the notations as in definition 1.2.3, we have the following properties:

a) For \( 1 \leq i \leq n \), \( \sigma_{\alpha_i}(\lambda_j) = \lambda_j - \delta_{ij} \alpha_i \).

b) For \( \lambda \in \Lambda \), \( \lambda = \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle \alpha_i > \lambda_i \).

c) Let \( \delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha \). Then \( \delta = \sum_{i=1}^{n} \lambda_i \), so \( \delta \) is a dominant weight.

**Proof.** a) This is direct from the definition of the \( \lambda_i \)'s and the definition of \( \sigma_{\alpha_j} \).

b) We have \( \langle \lambda - \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle \alpha_i, \alpha_j \rangle = \langle \lambda, \alpha_j \rangle - \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle \delta_{ij} = 0 \),

by definition of the \( \lambda_i \)'s and by linearity in the first variable of the product \( \langle \cdot, \cdot \rangle \).

This implies that for each \( \alpha_j \in \Delta \), we have \( \langle \lambda - \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle \alpha_i, \alpha_j \rangle = 0 \), and since \( \langle \cdot, \cdot \rangle \) is a scalar product we have \( \lambda = \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle > \lambda_i \).

c) By b), we have \( \delta = \sum_{i=1}^{n} \langle \delta, \alpha_i \rangle > \lambda_i \). But \( \langle \delta, \alpha_i \rangle = 1 \) for every \( i \), since \( \sigma_{\alpha_i}(\delta) = \delta - \langle \delta, \alpha_i \rangle \alpha_i \) by definition of \( \sigma_{\alpha_i} \), and we also have \( \sigma_{\alpha_i}(\delta) = \delta - \alpha_i \), since \( \sigma_{\alpha_i} \) permutes all the positive roots different from \( \alpha_i \). \( \square \)
1.2.2 Weights of an $L$-module

From now, all modules considered are assumed to be finite-dimensional.

Since it is equivalent to have a finite-dimensional representation of a semisimple algebra $L$ or a finite-dimensional $L$-module, we will work with the decomposition of the modules for understanding the representations. In order to find the decompositions into irreducible summands of the modules, we will use the weights of a module, as presented here.

**Definition 1.2.5.** Let $V$ be an $L$-module and $\lambda \in \mathfrak{h}^*$. We define $V_\lambda = \{v \in V \mid h.v = \lambda(h)v \ \forall h \in \mathfrak{h}\}$. If $V_\lambda \neq \{0\}$, we call $V_\lambda$ a weight space and we say that $\lambda$ is a weight of $V$. We denote by $\Lambda(V)$ the set of weights of $V$.

**Remark.** Since we are only working with finite-dimensional modules, we can check that this definition of a weight corresponds to the definition made in section 1.2.1, and so we can apply the results obtained there, as explained in section 21.1 from [Hum].

**Lemma 1.2.6.** Let $V$ be an $L$-module. Then $L_\alpha$ maps $V_\lambda$ into $V_{\lambda+\alpha}$ for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Phi$.

**Proof.** See section 20.1 from [Hum].

**Proposition 1.2.7.** Let $V$ be an $L$-module. Then $V$ is equal to the direct sum of its weight spaces.

**Proof.** See section 20.1 from [Hum].

**Definition 1.2.8.** Let $V$ be an $L$-module and $\lambda \in \Lambda(V)$. If $v^+ \in V_\lambda$ is a nonzero vector such that $e_\alpha.v^+ = 0$ for every $\alpha \in \Delta$, we say that $v^+$ is a maximal vector of weight $\lambda$.

**Lemma 1.2.9.** Let $v^+$ is a maximal vector of weight $\lambda \in \mathfrak{h}^*$ and $V = \mathfrak{U}(L).v^+$. Then the following hold:

1) The weights of $V$ are of the form $\mu = \lambda - \sum_{i=1}^n c_i \alpha_i$ with the $c_i$ nonnegative integers,

2) $\dim(V_\lambda) = 1$,

3) $V$ is an indecomposable $L$-module.

**Definition 1.2.10.** An $L$-module as defined in lemma 1.2.9 is a standard cyclic module with highest weight $\lambda$.

**Proof.** See section 20.2 from [Hum]

**Theorem 1.2.11.** Two irreducible standard cyclic $L$-modules with the same highest weight are isomorphic.

**Proof.** See section 20.3 from [Hum].

By Weyl’s theorem of complete reducibility (theorem 1.2.2, we will seek to determine the irreducible summands of the decomposition of a $L$-module. And by theorem 1.2.11, it will be enough to look for the highest weights of these summands.

In fact, a finite-dimensional $L$-module $V$ always has a highest weight, since there is only a finite number of weights by the condition on the dimensions, which implies that one of them has to be maximal. Moreover, if $\lambda$ is the maximal weight, a vector of weight $\lambda$ is necessarily a maximal vector, by lemma 1.2.6 and by maximality of $\lambda$. Consequently, a finite-dimensional irreducible $L$-module $V$ with highest weight $\lambda$ is necessarily standard cyclic, since the maximal vector of weight $\lambda$ generates a standard cyclic submodule, which is equal to $V$ by irreducibility.
Definition 1.2.12. A finite-dimensional irreducible $L$-module with highest weight $\lambda$ is denoted by $V_L(\lambda)$.

Proposition 1.2.13. Let $V_L(\lambda)$ be an irreducible $L$-module with highest weight $\lambda$. Then all the weights $\mu$ of $V_L(\lambda)$ are such that $\sigma(\mu) \prec \lambda$, for every $\sigma$ in the Weyl group.

Proof. See section 21.3 from [Hum]. \qed

1.3 Multiplicities and dimensions

In this section we will introduce the notion of multiplicity of a weight in a chosen module, and see how this is useful to find the irreducible summands of an $L$-module. We will also give some results about the dimensions of some specific modules.

1.3.1 General multiplicities

Definition 1.3.1. Let $V$ be an irreducible $L$-module. Let $\mu \in \Lambda(V)$. Then the multiplicity of $\mu$ in $V$ is $m_V(\mu) = \dim(V_{\mu})$. If $\mu \in \mathfrak{h}^*$ is not a weight of $V$, we say that $m_V(\mu) = 0$.

Theorem 1.3.2. (Freudenthal’s formula). Let $V_L(\lambda)$ be an irreducible $L$-module with highest weight $\lambda$. The multiplicity $m_{V_L(\lambda)}(\mu)$ of a weight $\mu$ in $V_L(\lambda)$ is given recursively by

$$((\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta)) m_{V_L(\lambda)}(\mu) = 2 \sum_{\alpha > 0} \sum_{i=1}^{\infty} m_{V_L(\lambda)}(\mu + i\alpha)(\mu + i\alpha, \alpha)$$

(1.1)

where $\delta$ denotes the half of the sum of the positive roots of $\Phi$, the root system of $L$.

Proof. See section 22.4 from [Hum]. \qed

We can also write Freudenthal’s formula as

$$\left(2(\lambda + \delta, \lambda - \mu) - |\lambda - \mu|^2\right) m_{V_L(\lambda)}(\mu) = 2 \sum_{\alpha > 0} \sum_{i=1}^{\infty} m_{V_L(\lambda)}(\mu + i\alpha)(\mu + i\alpha, \alpha),$$

which makes it easier to compute. This can be obtained directly by expanding the left-hand side of (1.1), using the fact that $(-, -)$ is a scalar product.

Proposition 1.3.3. Let $V_L(\lambda)$ be an irreducible $L$-module with highest weight $\lambda$. Let $\mu \in \Lambda(V_L(\lambda))$ and let $\sigma \in W$, with $W$ the Weyl group of the root system associated to $L$. Then

$$\dim(V_L(\lambda)_{\mu}) = \dim(V_L(\lambda)_{\sigma(\mu)}).$$

Proof. See section 21.2 from [Hum]. \qed

Note that in term of multiplicities, this means that $m_{V_L(\lambda)}(\mu) = m_{V_L(\lambda)}(\sigma(\mu))$ in $V_L(\lambda)$, for every $\sigma \in W$ and every weight $\mu$ of $V_L(\lambda)$.
1.3.2 Specific weight multiplicities

As usual, let $L$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $L$, $\Phi$ the corresponding root system. Fix a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of $\Phi$, and $\lambda_1, \ldots, \lambda_n$ the corresponding fundamental dominant weights.

We will require the following results which help us calculate certain weight multiplicities.

**Proposition 1.3.4.** Let $L$ be a semisimple Lie algebra, and $V_L(\lambda)$ be an irreducible $L$-module with highest weight $\lambda$. Write $\lambda = \sum_{r=1}^{n} a_r \lambda_r$. We define

$$\lambda_{i,x} = \sum_{r=1}^{i-1} a_r \lambda_r + x \lambda_i + \sum_{r=i+1}^{n} a_r \lambda_r$$

for every $1 \leq i \leq n$ and every integer $x$ such that $x \geq a_i$. Similarly, for $\mu = \lambda - \sum_{r=1}^{n} c_r \alpha_r$ such that $\mu < \lambda$, we define

$$\mu_{i,x} = \lambda_{i,x} - \sum_{r=1}^{n} c_r \alpha_r.$$

Suppose that there exists $1 \leq i \leq n$ such that

$$m_{V_L(\lambda_{i,x})}(\mu_{i,x} + j\alpha) = m_{V_L(\lambda_{i,y})}(\mu_{i,y} + j\alpha)$$

for every $x, y \geq a_i$, every positive integer $j$ and every $\alpha \in \Phi^+$. Then

$$m_{V_L(\lambda_{i,x})}(\mu_{i,x}) = m_{V_L(\lambda_{i,y})}(\mu_{i,y})$$

for every $x, y \geq a_i$.

**Proof.** Reference See proposition 1.3.4 from [Cav].

**Lemma 1.3.5.** Let $V_L(\lambda)$ be an irreducible $L$-module of highest weight $\lambda$. Suppose that $\mu = \lambda - \sum_{\alpha \in S} c_\alpha \alpha$ is a dominant weight, with $S$ a subset of the set of simple roots $\Delta$. Then $m_{V_L(\lambda)}(\mu) = m_{V'}(\mu')$, where $X$ is the subalgebra generated by $\{e_\alpha, f_\alpha \mid \alpha \in S\}$, $\mu' = \mu|_X$ and $V' = V_X(\lambda|_X)$.

**Proof.** See lemma 2.2.8 from [Bur].

The next result concerns a simple Lie algebra of type $A_n$.

Let $V_{A_n}(\lambda)$ be an irreducible representation of a Lie algebra of type $A_n$, with highest weight $\lambda$. Write $\lambda$ as $\sum_{i=1}^{m_1} a_i \lambda_i$, where $m_1$ (respectively $m_2$) denotes the first (respectively the last) $j$ such that $a_j \neq 0$. Then we can write $\lambda = \sum_{i=1}^{m_2-m_1+1} b_i \lambda_{i+m_1-1}$. For any integer $j$, we set $s_{2j-1}$ = cardinality of the $j^{th}$ pack of nonzero $b_i$’s and $s_{2j} = $ cardinality of the $j^{th}$ pack of zero $b_i$’s. Let $r$ be the greatest integer such that $s_r \neq 0$. Let $\mu = \lambda - \sum_{i=1}^{r} \alpha_i$, $I = \{1, \ldots, \frac{r-1}{2}\}$ and $J = \{1, \ldots, \frac{r-1}{2}\}$.

So we are in the following case :

---

![Diagram](image)

---

**Lemma 1.3.6.** With the previous notation, the multiplicity of $\mu$ in $V_{A_n}(\lambda)$ is given by

$$m_{V_{A_n}(\lambda)}(\mu) = 2 \sum_{i \in I} (s_{2i-1}) \prod_{j \in J} (s_{2j} + 2).$$

**Proof.** Reference See ?? from [Cav].
1.3.3 Dimensions

We present here some formulas that will be useful, since they allow us to calculate the dimension of an irreducible $L$-module, knowing its highest weight. We obtain them by Weyl’s degree formula:

**Proposition 1.3.7. (Weyl’s degree formula).** Let $\lambda$ be a dominant weight, and $V_L(\lambda)$ the irreducible $L$-module with highest weight $\lambda$. Then

$$\dim(V_L(\lambda)) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \delta, \alpha)}{\prod_{\alpha \in \Phi^+} (\delta, \alpha)},$$

where $\delta$ denotes the half of the sum of the positive roots.

**Proof.** See section 24.3 from [Hum].

We give here the formulas to compute the dimensions of an irreducible $L$-module, knowing its highest weight, when $L$ is of type $A_2$, $B_2$, $B_3$, $D_3$, $D_4$ or $G_2$. The calculations can be carried out using the data presented in appendix A.

<table>
<thead>
<tr>
<th>Type</th>
<th>Weight $\lambda$</th>
<th>Dimension of $V(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>$\lambda = a\lambda_1 + b\lambda_2$</td>
<td>$\dim(V_{A_2}(\lambda)) = \frac{1}{2}(a+1)(b+1)(a+b+1)$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\lambda = a\lambda_1 + b\lambda_2$</td>
<td>$\dim(V_{B_2}(\lambda)) = \frac{1}{72}(a+1)(b+1)(c+1)$ $\times(a+b+2)(b+c+2)(2b+c+3)$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3$</td>
<td>$\dim(V_{B_3}(\lambda)) = \frac{1}{720}(a+1)(b+1)(c+1)$ $\times(a+b+2)(b+c+2)(2b+c+3)$ $\times(a+b+c+3)(a+2b+c+4)(2a+2b+c+5)$</td>
</tr>
<tr>
<td>$D_3$</td>
<td>$\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3$</td>
<td>$\dim(V_{D_3}(\lambda)) = \frac{1}{12}(a+1)(b+1)(c+1)$ $\times(a+b+2)(a+c+2)(a+b+c+3)$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$\lambda = a\lambda_1 + b\lambda_2$ $\quad+$ $c\lambda_3 + d\lambda_4$</td>
<td>$\dim(V_{D_4}(\lambda)) = \frac{1}{480}(a+1)(b+1)(c+1)(d+1)$ $\times(a+b+2)(b+c+2)(b+d+2)$ $\times(a+b+c+3)(a+b+d+3)(b+c+d+3)$ $\times(a+b+c+d+4)(a+2b+c+d+5)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\lambda = a\lambda_1 + b\lambda_2$</td>
<td>$\dim(V_{G_2}(\lambda)) = \frac{1}{120}(a+1)(b+1)(a+b+2)$ $\times(a+2b+3)(a+3b+4)(2a+3b+5)$</td>
</tr>
</tbody>
</table>

Table 1.2: Dimensions for representations of small rank simple Lie algebras

**Proposition 1.3.8.** Let $L$ be a semisimple Lie algebra with root system generated by the simple roots $\alpha_1, \ldots, \alpha_n$. Let $X$ be a semisimple subalgebra of $L$ generated by $e_{\alpha_i}, f_{\alpha_i}$ for $2 \leq i \leq n$. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding dominant fundamental weights of $L$ and $\mu_2, \ldots, \mu_n$ the corresponding fundamental dominant weights of $X$. Let $\delta_L = \sum_{i=1}^n \lambda_i$ and $\delta_X = \sum_{i=2}^n \mu_i$. Let $\lambda = \sum_{i=1}^n b_i \lambda_i$ and $\mu = \sum_{i=2}^n b_i \mu_i$ be dominant weights for $L$, respectively $X$. Then we have

$$\dim(V_L(\lambda)) = \left(\prod_{\alpha \in \Phi^+_L} \frac{<\lambda + \delta_L, \alpha>} {<\delta_L, \alpha>}\right) \dim(V_X(\mu)),$$
with $\Phi_1^+ = \{\sum_{i=1}^{n} a_i \alpha_i > 0 \mid a_1 \neq 0\}$.

**Proof.** By Weyl’s degree formula, we have

$$\dim(V_L(\lambda)) = \prod_{\alpha > 0} \frac{\langle \lambda + \delta_L, \alpha \rangle}{\langle \delta_L, \alpha \rangle}.$$

Let $\Phi^+$ be the set of positive roots for $L$. Then we can write it as the disjoint union $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, with

$$\Phi_0^+ = \{\sum_{i=1}^{n} a_i \alpha_i \in \Phi^+ \mid a_1 = 0\},$$

and

$$\Phi_1^+ = \{\sum_{i=1}^{n} a_i \alpha_i \in \Phi^+ \mid a_1 \neq 0\}.$$

Let $\Phi_X^+$ be the set of positive roots for $X$. By definition of $X$, we have $\Phi_X^+ = \Phi_0^+$. And since $\delta_L$ is the sum of the fundamental weights for $L$ and $\langle \lambda, \alpha \rangle = \delta_{ij}$, we obtain that for each $\alpha \in \Phi_X^+$, we have $\langle \delta_X, \alpha \rangle = \langle \delta_L, \alpha \rangle$ and $\langle \mu + \delta_X, \alpha \rangle = \langle \lambda + \delta_L, \alpha \rangle$.

We summarize this using Weyl’s degree formula, and we have

$$\dim(V_L(\lambda)) = \prod_{\alpha > 0} \frac{\langle \lambda + \delta_L, \alpha \rangle}{\langle \delta_L, \alpha \rangle},$$

and

$$= \left( \prod_{\alpha \in \Phi_1^+} \frac{\langle \lambda + \delta_L, \alpha \rangle}{\langle \delta_L, \alpha \rangle} \right) \left( \prod_{\alpha \in \Phi_0^+} \frac{\langle \lambda + \delta_L, \alpha \rangle}{\langle \delta_L, \alpha \rangle} \right) \left( \prod_{\alpha \in \Phi_X^+} \frac{\langle \mu + \delta_X, \alpha \rangle}{\langle \delta_X, \alpha \rangle} \right) \dim(V_X(\mu)).$$

We introduce now some results about the dimensions of representations of $D_{n+1}$ and of representations of $B_n$, that we will need later, in the proof of 4.2.1. We will denote by $\lambda^{(n)}_i$ the fundamental weights of $D_n$ and by $\mu^{(n)}_i$ the fundamental weights of $B_n$.

**Lemma 1.3.9.** For every $b \in \mathbb{N}$, we have the equality

$$\dim(V_{D_{n+1}}(b\lambda^{(n+1)})) = \dim(V_{B_n}(b\mu^{(n)}_i)).$$

**Proof.** We work by induction on the rank. In fact, we know that $\dim(V_{D_2}(b\lambda_2)) = \dim(V_{B_2}(b\mu_2))$ and $\dim(V_{D_3}(b\lambda_3)) = \dim(V_{B_3}(b\mu_3))$ for every $b \in \mathbb{N}$, by the table 1.2.

Now we suppose that this equality holds for $V_{D_{m+1}}(b\lambda_m)$ and $V_{B_m}(b\mu_m)$ for every $m < n$ and we will show by induction that this is true for $n$. Proposition 1.3.8 allows us to compute the dimension of $V_{D_{n+1}}(b\lambda^{(n+1)}_n)$ as a multiple of $\dim(V_{D_n}(b\lambda^{(n)}_{n-1}))$. Similarly, we will express $\dim(V_{B_n}(b\mu^{(n)}_n))$ as a multiple of $\dim(V_{B_{n-1}}(b\mu^{(n-1)}_{n-1}))$, and then apply our induction hypothesis.
Here we obtain

\[
\dim(V_{D_{n+1}}(b\lambda_{(n+1)}^{n+1})) = \left(\prod_{i=n}^{2n-1} \left(\frac{b+i}{i}\right)\right) \dim(V_{D_{n}}(b\lambda_{(n)}^{n}))
\]

and

\[
\dim(V_{B_{n}}(b\mu_{(n)}^{n})) = \left(\prod_{i=n}^{2n-1} \left(\frac{b+i}{i}\right)\right) \dim(V_{B_{n-1}}(b\mu_{(n-1)}^{n-1})).
\]

By our induction hypothesis, we conclude that \(\dim(V_{D_{n+1}}(b\lambda_{(n+1)}^{n+1})) = \dim(V_{B_{n}}(b\mu_{(n)}^{n}))\).

\[\Box\]

**Lemma 1.3.10.** For \(b \in \mathbb{N}\), we have

\[
\dim(V_{D_{n+1}}(\lambda_{1}^{(n+1)} + b\lambda_{(n+1)}^{n+1})) = \dim(V_{B_{n}}(\mu_{1}^{(n)} + b\mu_{(n)}^{n})) + \dim(V_{B_{n}}(b\mu_{(n)}^{n}))
\]

**Proof.** We use proposition 1.3.8, and we get

\[
\dim(V_{D_{n+1}}(\lambda_{1}^{(n+1)} + b\lambda_{(n+1)}^{n+1})) = \left(\prod_{i=n}^{2n-1} \left(\frac{b+i+1}{i}\right)\right) (n+1) \dim(V_{D_{n}}(b\lambda_{(n)}^{n}))
\]

\[
\dim(V_{B_{n}}(\mu_{1}^{(n)} + b\mu_{(n)}^{n})) = \left(\prod_{i=n+1}^{2n-1} \left(\frac{b+i}{i}\right)\right) (b+2n+1) \dim(V_{B_{n-1}}(b\mu_{(n-1)}^{n-1}))
\]

\[
\dim(V_{B_{n}}(b\mu_{(n)}^{n})) = \left(\prod_{i=n}^{2n-1} \left(\frac{b+i}{i}\right)\right) \dim(V_{B_{n-1}}(b\mu_{(n-1)}^{n-1})).
\]

By the previous lemma, we know that \(\dim(V_{D_{n}}(b\lambda_{(n)}^{n})) = \dim(V_{B_{n-1}}(b\mu_{(n-1)}^{n-1}))\), and we get

\[
\dim(V_{B_{n}}(\mu_{1}^{(n)} + b\mu_{(n)}^{n})) + \dim(V_{B_{n}}(b\mu_{(n)}^{n}))
\]

\[
= \left(\prod_{i=n+1}^{2n-1} \left(\frac{b+i}{i}\right)\right) (b+2n+1 + \frac{b+n}{n}) \dim(V_{D_{n}}(b\lambda_{(n)}^{n}))
\]

\[
= \left(\prod_{i=n+1}^{2n-1} \left(\frac{b+i}{i}\right)\right) \frac{(n+1)(b+2n)}{n} \dim(V_{D_{n}}(b\lambda_{(n)}^{n}))
\]

\[
= \left(\prod_{i=n}^{2n-1} \left(\frac{b+i+1}{i}\right)\right) (n+1) \dim(V_{D_{n}}(b\lambda_{(n)}^{n}))
\]

\[
= \dim(V_{D_{n+1}}(\lambda_{1}^{(n+1)} + b\lambda_{(n+1)}^{n+1})).
\]

\[\Box\]

**Lemma 1.3.11.** For every \(1 < i < n\) and every \(b \in \mathbb{N}\), we have

\[
\dim(V_{D_{n+1}}(\lambda_{i}^{(n+1)} + b\lambda_{(n+1)}^{n+1})) = \dim(V_{B_{n}}(\mu_{i}^{(n)} + b\mu_{(n)}^{n})) + \dim(V_{B_{n}}(\mu_{i-1}^{(n)} + b\mu_{(n)}^{n})).
\]
Similarly, replacing $V$ and we can express it as
\[ \dim(V_{D_{n+i}}(\lambda^{(n+1)}_i + b\lambda^{(n+1)}_i)) = \dim(V_{B_{n-i}}(b\lambda^{(n-i)}_n)). \]

By induction, we get the following equality:
\[ \dim(V_{D_{n+i}}(\lambda^{(n+1)}_i + b\lambda^{(n+1)}_i)) = \prod_{k=1}^i \left( \frac{n-i+k+1}{k} \right) \prod_{j=n-i+k}^{2n-2i+k} \left( \frac{b+j+1}{j} \right) \times \prod_{k=2}^{i} \left( \prod_{m=k}^{2k-2} \left( \frac{b+2n-2i+m+3}{2n-2i+m+1} \right) \right) \dim(V_{D_{n+i+1}}(b\lambda^{(n-i+1)}_n)). \]

For $V_{B_n}(\mu^{(n)}_i + b\mu^{(n)}_n)$, we have
\[ \dim(V_{B_n}(\mu^{(n)}_i + b\mu^{(n)}_n)) = \prod_{k=1}^i \left( \frac{n-i+k}{k} \right) \prod_{j=n-i+k}^{2n-2i+k} \left( \frac{b+j+1}{j} \right) \times \prod_{k=2}^{i} \left( \prod_{m=k}^{2k-2} \left( \frac{b+2n-2i+m+3}{2n-2i+m+1} \right) \right) \prod_{k=2}^{i} \left( \frac{2n-2i+2k-1}{2n-2i+k} \right) \frac{b+2n-2i+k+2}{b+2n-2i+2k+1} \dim(V_{B_{n-i}}(b\mu^{(n-i)}_{n-i})), \]

which is equal to
\[ \dim(V_{B_n}(\mu^{(n)}_i + b\mu^{(n)}_n)) = \prod_{k=1}^i \left( \frac{n-i+k+1}{k} \right) \left( \frac{n-i+1}{n+1} \right) \prod_{j=n-i+k}^{2n-2i+k} \left( \frac{b+j+1}{j} \right) \times \prod_{k=2}^{i} \left( \prod_{m=k}^{2k-2} \left( \frac{b+2n-2i+m+3}{2n-2i+m+1} \right) \right) \prod_{k=2}^{i} \left( \frac{2n-2i+2k-1}{2n-2i+k} \right) \frac{b+2n-2i+k+2}{b+2n-2i+2k+1} \dim(V_{B_{n-i}}(b\mu^{(n-i)}_{n-i})), \]

and we can express it as
\[ \dim(V_{B_n}(\mu^{(n)}_i + b\mu^{(n)}_n)) = \left( \frac{n-i+1}{n+1} \right) \left( \frac{2n-2i+1}{b+2n-2i+2} \right) \frac{b+2n-2i+3}{2n-2i+1} \times \prod_{k=2}^{i} \left( \frac{b+2n-2i+k+2}{b+2n-2i+k+1} \right) \dim(V_{D_{n+i}}(\lambda^{(n+1)}_i + b\lambda^{(n+1)}_i)) \]
\[ = \left( \frac{n-i+1}{n+1} \right) \left( \frac{b+2n-i+2}{b+2n-2i-2} \right) \dim(V_{D_{n+i}}(\lambda^{(n+1)}_i + b\lambda^{(n+1)}_i)). \]

Similarly, replacing $i$ by $i-1$, we have
\[ \dim(V_{B_n}(\mu^{(n)}_{i-1} + b\mu^{(n)}_n)) = \prod_{k=1}^{i-1} \left( \frac{n-i+k+1}{k} \right) \prod_{j=n-i+k+1}^{2n-2i+k+2} \left( \frac{b+j+1}{j} \right) \times \prod_{k=1}^{i-1} \left( \prod_{m=k}^{2k-1} \left( \frac{b+2n-2i+m+4}{2n-2i+m+2} \right) \right) \prod_{k=1}^{i-1} \left( \frac{2n-2i+k+2}{b+2n-2i+k+3} \right) \dim(V_{B_{n-i+1}}(b\mu^{(n-i+1)}_{n-i})). \]
By repeatedly applying proposition 1.3.8, we get

\[
= \prod_{k=1}^{i} \left( \frac{n-i+k+1}{k} \right) \prod_{k=2}^{i} \left( \frac{2n-2i+k+1}{j} \right) \prod_{k=2}^{2k-2} \frac{b+2n-2i+m+3}{2n-2i+m+1} \prod_{k=2}^{i} \frac{2n-2i+k+1}{b+2n-2i+k+2} \dim(V_{B_{n-i+1}}(b\mu_{n-i}^{(n-i+1)}))
\]

By the proof of lemma 1.3.9, we know that

\[
\dim(V_{B_{n-i+1}}(b\mu_{n-i}^{(n-i+1)})) = \prod_{j=n-i+1}^{2n-2i+1} \left( \frac{b+j}{j} \right) \dim(V_{B_{n-i}}(b\mu_{n-i-1}^{(n-i)}))
\]

so we obtain

\[
\dim(V_{B_{n}}(\mu_{i-1}^{(n)} + b\mu_{n}^{(n)})) = \left( \frac{i}{n+1} \right) \left( \frac{b+n-i+1}{b+2n-2i+1} \right) \prod_{k=2}^{i} \left( \frac{b+2n-2i+k+2}{2n-2i+k+1} \right) \prod_{k=2}^{2n-2i+1} \frac{b+2n-2i+k+1}{b+2n-2i+k+2} \dim(V_{D_{n-i+1}}(\lambda_{i}^{(n+1)} + b\lambda_{n}^{(n+1)}))
\]

In conclusion, we have

\[
\dim(V_{B_{n}}(\mu_{i}^{(n)} + b\mu_{n}^{(n)})) + \dim(V_{B_{n}}(\mu_{i-1}^{(n)} + b\mu_{n}^{(n)})) = \left( \frac{n-i+1}{n+1} \right) \left( \frac{b+2n-i+2}{b+2n-2i-2} \right) \dim(V_{D_{n-i+1}}(\lambda_{i}^{(n+1)} + b\lambda_{n}^{(n+1)})) + \left( \frac{i}{n+1} \right) \left( \frac{b+n-i+1}{b+2n-2i+1} \right) \dim(V_{D_{n-i+1}}(\lambda_{i}^{(n+1)} + b\lambda_{n}^{(n+1)})) = \dim(V_{D_{n-i+1}}(\lambda_{i}^{(n+1)} + b\lambda_{n}^{(n+1)})),
\]

which is the desired equality.

**Lemma 1.3.12.** For \( b \in \mathbb{N} \), we have the equality

\[
\dim(V_{D_{n+1}}(b\lambda_{n}^{(n+1)} + \lambda_{n+1}^{(n+1)})) = \dim(V_{B_{n}}((b+1)\mu_{n}^{(n)})) + \dim(V_{B_{n}}(\mu_{n-1}^{(n)} + (b-1)\mu_{n}^{(n)})).
\]

**Proof.** First, we obtain

\[
\dim(V_{D_{n+1}}(b\lambda_{n}^{(n+1)} + \lambda_{n+1}^{(n+1)})) = \frac{(b+n)(n+1)}{(b+n+1)n} \left( \prod_{i=n}^{2n-1} \left( \frac{b+i+1}{i} \right) \right) \dim(V_{D_{n}}(b\lambda_{n-1}^{(n)} + \lambda_{n}^{(n)})).
\]

By repeatedly applying proposition 1.3.8, we get

\[
\dim(V_{D_{n+1}}(b\lambda_{n}^{(n+1)} + \lambda_{n+1}^{(n+1)})) = \prod_{k=3}^{n} \left( \frac{b+k+1}{b+k} \right) ^{2k-1} \left( \prod_{i=k}^{2k-1} \left( \frac{b+i+1}{i} \right) \right) \dim(V_{D_{n}}(b\lambda_{2}^{(3)} + \lambda_{3}^{(3)})).
\]
Similarly, we have 

$$\dim(V_{D_{n+1}}(b\lambda_n \mu_{n+1})) = \frac{n + 1}{b + n + 1} \cdot \frac{(b + 1)(b + 2)(b + 3)(b + 4)}{6} y.$$ 

We set 

$$y = \left(\prod_{k=3}^{n} \dim\left(\prod_{i=k}^{2k-1} \left(\frac{b + i + 1}{i}\right)\right)\right)$$

and we have

$$\dim(V_{D_{n+1}}(b\lambda_n \mu_{n+1} + \lambda_{n+1})) = \frac{n + 1}{b + n + 1} \cdot \frac{(b + 1)(b + 2)(b + 3)(b + 4)}{6} y.$$ 

Similarly, we have

$$\dim(V_{B_n}((b + 1)\mu_{n+1})) = \left(\prod_{i=n}^{2n-1} \left(\frac{b + i + 1}{i}\right)\right) \dim(V_{B_{n-1}}((b + 1)\lambda_{n-1}))$$

$$= \left(\prod_{k=3}^{n} \left(\prod_{i=k}^{2k-1} \left(\frac{b + i + 1}{i}\right)\right)\right) \dim(V_{B_2}((b + 1)\lambda_2))$$

$$= \frac{(b + 2)(b + 3)(b + 4)}{6} y.$$ 

And finally we compute

$$\dim(V_{B_n}(\mu_{n+1} + (b - 1)\mu_{n+1})) = \frac{(b + 1)n}{(b + n + 1)(n - 1)} \left(\prod_{i=n}^{2n-1} \left(\frac{b + i + 1}{i}\right)\right)$$

$$\times \dim(V_{B_{n-1}}(\mu_{n+1} - (b - 1)\mu_{n+1}))$$

$$= \prod_{k=3}^{n} \left(\prod_{i=k}^{2k-1} \left(\frac{b + i + 1}{i}\right)\right) \dim(V_{B_2}(\mu_2 + (b - 1)\mu_2))$$

$$= \left(\prod_{k=3}^{n} \left(\prod_{i=k}^{2k-1} \left(\frac{b + i + 1}{i}\right)\right)\right) \frac{n(b + 3)}{2(b + n + 1)} \dim(V_{B_2}(\mu_2 + (b - 1)\mu_2))$$

$$= \left(\prod_{k=3}^{n} \left(\prod_{i=n}^{2n-1} \left(\frac{b + i + 1}{i}\right)\right)\right) \frac{n(b + 2)(b + 3)(b + 4)}{b + n + 1} \cdot \frac{6}{y}$$

In the end, we obtain the sum of the dimensions as follows:

$$\dim(V_{B_n}((b + 1)\mu_{n+1})) + \dim(V_{B_n}(\mu_{n+1} + (b - 1)\mu_{n+1})) = \frac{(b + 2)(b + 3)(b + 4)}{6} \left(1 + \frac{bn}{b + n + 1}\right) y$$

$$= \frac{n + 1}{b + n + 1} \cdot \frac{(b + 1)(b + 2)(b + 3)(b + 4)}{6} y = \dim(V_{D_{n+1}}(b\lambda_n + \lambda_{n+1})).$$

\(\square\)
Chapter 2

More advanced topics

2.1 Restriction of a representation to a subalgebra

Let $Y$ be a simple Lie algebra, $X$ be a simple subalgebra of $Y$, and $V_Y$ be an irreducible $Y$-module. We will explain the method that we use to determine the restriction of $V_Y$ to $X$. The embedding of $X$ into $Y$ will be fixed at the beginning of each case.

Let $\mathfrak{h}_X$ be a Cartan subalgebra of $X$, with $\Phi_X$ the associated root system and $\Delta_X$ a base of $\Phi_X$. Let $\mathfrak{b}_X$ be the standard corresponding Borel subalgebra. Then there exists a Cartan subalgebra $\mathfrak{h}_Y$ of $Y$ such that $\mathfrak{h}_X \subset \mathfrak{h}_Y$, by the characterisation of the Cartan subalgebras as the maximal toral subalgebras made in theorem 1.1.29. Let $\Phi_Y$ be the root system associated to $\mathfrak{h}_Y$ with base $\Delta_Y$.

Since $\mathfrak{b}_X$ is a solvable subalgebra of $X$, it is also a solvable subalgebra of $Y$ so there exists a Borel subalgebra $\mathfrak{b}_Y$ of $Y$ such that $\mathfrak{b}_X \subset \mathfrak{b}_Y$. By theorems 1.1.30 and 1.1.32, we can assume that $\mathfrak{b}_Y$ is the standard Borel subalgebra of $Y$ corresponding to $\mathfrak{h}_Y$.

By these inclusions, every weight $\omega \in \mathfrak{h}_Y^*$ can be viewed as an element of $\mathfrak{h}_X^*$, when restricted to $X$. We will then denote the restriction of $\omega$ to $\mathfrak{h}_X$ by $\omega|_X$. We also denote by $\alpha_i$ the simple roots of the algebra $Y$ and by $\beta_i$ the simple roots of the subalgebra $X$.

We will explain the method used when seeking to determine the irreducible representations of $Y$ which decompose into exactly two irreducible summands upon restriction to the subalgebra $X$, and introduce some general results that we will need for the study of these restrictions.

**Lemma 2.1.1.** Let $V_Y(\lambda)$ be an irreducible $Y$-module with highest weight $\lambda$, and $v^+$ be a maximal vector of $V_Y(\lambda)$ with respect to the Borel subalgebra $\mathfrak{b}_Y$ of $Y$, of weight $\lambda$. Then $v^+$ is a maximal vector for the restriction $(V_Y(\lambda))|_X$, of weight $\lambda|_X$.

**Proof.** Since the Borel subalgebra of $X$ will be chosen such that it is included in the Borel subalgebra of $Y$ by the embedding, and a vector is maximal for $Y$ if each $Y_\alpha$ with $\alpha > 0$ acts on it trivially, and is a weight vector for the Cartan subalgebra $\mathfrak{h}_Y$, then $v^+$ is a maximal vector for the restriction $(V_Y(\lambda))|_X$. The fact that $v^+$ has weight $\lambda|_X$ in the restriction is direct since $v^+$ has weight $\lambda$ in $V_Y(\lambda)$. □

The above lemma shows that one of the irreducible summands of the restriction of $V_Y(\lambda)$ to a representation of $X$ has $v^+$ as a maximal vector. Then $V_X(\lambda|_X)$ is a first irreducible summand of the restriction $(V_Y(\lambda))|_X$.
Lemma 2.1.2. Let $V_Y(\lambda)$ be an irreducible representation of $Y$ with highest weight $\lambda$. Let $\mu$ be a weight of $(V_Y(\lambda))|_{X}$. Suppose that

$$m_{\nu_S(\lambda|_{X})}(\mu) < \sum_{\nu \in \Lambda(V_Y(\lambda)), \nu|_{X} = \mu} m_{V_Y(\lambda)}(\nu).$$

Then $\mu$ appears in an irreducible summand of the restriction $(V_Y(\lambda))|_{X}$ different than $V_X(\lambda|_{X})$.

Proof. By theorem 1.2.2, we can write $V_Y(\lambda)|_{X} = V_X(\lambda|_{X}) \oplus W$, for $W$ an $X$-module. By the hypothesis, the multiplicity of $\mu$ in $V_X(\lambda|_{X})$ is strictly smaller than the multiplicity of $\mu$ in $V_Y(\lambda)|_{X}$, which implies that $\mu$ must be a weight of $W$, and that $W \neq \{0\}$, so there is a second irreducible summand.

Remark. The lemma as stated gives a way to detect the existence of a second irreducible summand. When we already know two irreducible summands $V_X(\lambda|_{X})$ and $V_X(\omega)$ of the restriction of $V_Y(\lambda)$ to $X$, we can generalize this lemma to establish the existence of a third irreducible summand, namely if $\mu$ is a weight of $(V_Y(\lambda))|_{X}$ such that

$$m_{\nu_S(\lambda|_{X})}(\mu) + m_{\nu_S(\omega)}(\mu) < \sum_{\nu \in \Lambda(V_Y(\lambda)), \nu|_{X} = \mu} m_{V_Y(\lambda)}(\nu),$$

then $\mu$ is also the weight of a third irreducible summand, distinct from $V_X(\lambda|_{X})$ and $V_X(\omega)$, by the same argument.

We observe that lemma 2.1.2 gives only a way to find a weight of another irreducible summand, but it does not help directly to determine this summand. When we want to find in which irreducible summand our weight appears, we need something more, namely the next lemma.

Since we will be interested in the case when we have exactly two irreducible summands, if we have already found two irreducible summands, it will be enough to see that there is a weight that must appear in another summand.

Lemma 2.1.3. Let $V_Y(\lambda)$ be an irreducible representation of $Y$ with highest weight $\lambda$. Set $\mu = \lambda|_{X}$. Let $\nu$ be a dominant weight of $V_Y(\lambda)|_{X}$ such that

$$\sum_{\gamma \in \Lambda(V_Y(\lambda)), \gamma|_{X} = \nu} m_{V_Y(\lambda)}(\gamma) > m_{\nu_S(\lambda)}(\nu).$$

Suppose that for every $\nu < \tau < \mu$, $\tau \neq \nu$, with $\tau \in \Lambda(V_X(\mu))$ dominant, we have

$$\sum_{\omega \in \Lambda(V_Y(\lambda)), \omega|_{X} = \tau} m_{V_Y(\lambda)}(\omega) = m_{\nu_S(\lambda)}(\tau).$$

Then $V_X(\nu)$ is an irreducible summand of $(V_Y(\lambda))|_{X}$.

Proof. By hypothesis on $\nu$ and by lemma 2.1.2, we know that there is another irreducible summand that appears in the decomposition of $(V_Y(\lambda))|_{X}$, and that $\nu$ must be a weight of this summand. Denote this summand by $V_X(\omega)$, with $\omega \neq \mu$. Then since $\nu$ is a weight of the summand, we have by proposition 1.2.13 that $\nu < \omega$. But if $\omega \neq \nu$, we know that $\omega$ cannot appear in a new summand since it has the same multiplicity in $V_Y(\lambda)$ and in $V_X(\mu)$. So this forces $\omega = \nu$, meaning that $V_X(\nu|_{X})$ is an irreducible summand of the restriction of $V_Y(\lambda)$ to $X$.

If we can find at least two highest weights distinct from $\lambda|_{X}$, we conclude that there are more than two irreducible summands in the restriction of the representation $V_Y$ into a representation of $X$. If we only manage to find one further highest weight, we check if the sum of the dimensions of the two summands is equal to the dimension of $V_Y$ or not.
2.2 Induction

2.2.1 The ideal \( I_J \) and the semisimple algebra \( \alpha' \)

Let \( L \) be a simple Lie algebra. As usual, we denote by \( K \) the base field. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( L \), and \( \Phi \) be the corresponding root system of \( L \). Let \( \Delta \) be a set of simple roots of \( L \), \( J \) be a subset of \( \Delta \), and set \( \Phi_J = (\sum_{\alpha \in J} \mathbb{Z}) \cap \Phi \).

We set \( a = \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle + \mathfrak{h} \) and \( I_J = \langle e_\alpha \mid \alpha \in \Phi^+ \Phi_J \rangle \), where \( \langle \ldots \rangle \) denotes the subalgebra of \( L \) generated by the elements. Then we set \( P_J = a \oplus I_J \).

**Lemma 2.2.1.** We can express \( I_J \) as \( \sum_{\alpha \in \Phi^+ \Phi_J} L_\alpha \).

*Proof.* It comes from the observation that the bracket of two elements of \( I_J \) is already in the linear span of \( \{e_\alpha \mid \alpha \in \Phi^+ \Phi_J\} \). \( \square \)

**Lemma 2.2.2.** With the previous definitions, \( P_J \) is a subalgebra of \( L \).

*Proof.* This comes by an easy calculation, showing that for \( p, q \in P_J \) we have \([p, q] \in P_J \). Using the fact that \( \Phi \) and \( \Phi_J \) are root systems, we see that for \( \alpha \in J \) and \( \beta \in \Phi^+ \Phi_J \), we have \( \alpha + \beta \in \Phi^+ \Phi_J \) if \( \alpha + \beta \) is a root, and \([L_\alpha, L_\beta] = \{0\}\) if \( \alpha + \beta \) is not a root. We also use the relations between the elements \( e_\alpha, f_\alpha \) and \( h_\alpha \) stated in proposition 1.1.41 to compute \([p, q]\), and we obtain the result. \( \square \)

**Lemma 2.2.3.** With these definitions, \( I_J \) is an ideal of \( P_J \) consisting of nilpotent elements.

*Proof.* We will show that \( I_J \) is an ideal of \( P_J \), i.e., that for every \( i \in I_J \) and every \( p \in P_J \), we have \([i, p] \in I_J \). Since \( i \in I_J \), we can express it as

\[
i = \sum_{\gamma \in \Phi^+ \Phi_J} d_\gamma e_\gamma, \text{ with } d_\gamma \in K.
\]

Similarly, the element \( p \) can be written as

\[
p = \sum_{\beta \in \Phi^+ \Phi_J} a_\beta e_\beta + \sum_{\alpha \in \Phi^+ J} b_\alpha e_\alpha + \sum_{\alpha \in \Phi^+ J} c_\alpha f_\alpha + h, \text{ with } a_\beta, b_\alpha, c_\alpha \in K \text{ and } h \in \mathfrak{h}.
\]

The following relations give the result:

1) \( [e_\gamma, e_\beta] = re_{\gamma+\beta} \in I_J \) with \( \gamma, \beta \in \Phi^+ \Phi_J \) if \( \gamma + \beta \) is a root, so \( \gamma + \beta \in \Phi^+ \Phi_J \),

2) \( [e_\gamma, e_\beta] = 0 \) if \( \gamma + \beta \) is not a root,

3) \( [e_\gamma, e_\alpha] = re_{\gamma+\alpha} \in I_J \) with \( \gamma \in \Phi^+ \Phi_J, \alpha \in \Phi^+_J \) if \( \gamma + \alpha \) is a root, which implies \( \gamma + \alpha \in \Phi^+ \Phi_J \),

4) \( [e_\gamma, f_\alpha] = 0 \) with \( \gamma \in \Phi^+ \Phi_J, \alpha \in \Phi^+_J \) if \( \gamma - \alpha \) is not a root,

5) \( [e_\gamma, f_\alpha] = re_{\gamma-\alpha} \) with \( \gamma \in \Phi^+ \Phi_J, \alpha \in \Phi^+_J \) if \( \gamma - \alpha \) is a root, which means that \( \gamma - \alpha \in \Phi^+ \Phi_J \),

6) \( [e_\gamma, h] = re_\gamma \in I_J \) with \( \gamma \in \Phi^+ \Phi_J, h \in \mathfrak{h} \).
Now we just need to show that $I_J$ consists of nilpotent elements, so equivalently we will show that for each $\alpha \in \Phi^+ \setminus \Phi_J$, $ad_{e_\alpha}$ is a nilpotent morphism, by lemma 1.1.7. But we deduce it directly from the relation between the generators of $L$ presented in proposition 1.1.41 and from lemma 1.1.12.

Lemma 2.2.4. Let $a' = [a, a]$. Then we have $a' = \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle$.

Proof. This comes from the fact that for any $h \in \mathfrak{h}$, for any $\alpha \in \Phi$, we know that $[e_\alpha, h]$ is a nonzero multiple of $e_\alpha$ and similarly $[f_\alpha, h]$ is a multiple of $f_\alpha$, by proposition 1.1.41. Using also that the bracket of two elements of $\mathfrak{h}$ is equal to zero, we have $[a, a] \subset \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle$.

We show then that $e_\alpha, f_\alpha \in [a, a]$ for every $\alpha \in J$, which will show the reverse inclusion. We know that $h_\alpha \in \mathfrak{h} \subset a$, and $e_\alpha, f_\alpha \in a$. Then we obtain

$$[e_\alpha, h_\alpha] = \langle \alpha, \alpha > e_\alpha = 2e_\alpha$$

and

$$[f_\alpha, h_\alpha] = -\langle \alpha, \alpha > f_\alpha = -2f_\alpha.$$ 

We have $2e_\alpha, -2f_\alpha \in [a, a]$, and since this is a subalgebra we deduce that $e_\alpha, f_\alpha \in [a, a]$, as desired, which shows the equality $a' = \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle$.

Definition 2.2.5. A subalgebra $H$ of a Lie algebra $L$ is reductive in $L$ if the adjoint representation of $H$ in $L$ is semisimple, i.e., if the module corresponding to $x \rightarrow ad_x \in \mathfrak{gl}(L)$ for $x \in H$ is completely reducible.

Proposition 2.2.6. Let $\mathfrak{k}$ be a subalgebra of $L$ such that $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$, and let $\mathfrak{h}' \subset \mathfrak{h}$, $P \subset \Phi$ such that $\mathfrak{k} = \mathfrak{h}' + \sum_{\alpha \in P} L_\alpha$. Then $\mathfrak{k}$ is reductive in $L$ if and only if $P = -P$.

Proof. See 20.7.5 from [Tau].

Lemma 2.2.7. The subalgebra $\mathfrak{a} = \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle + \mathfrak{h}$ is reductive in $L$.

Proof. Clearly, we have $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$. We can write $\mathfrak{a}$ as $\mathfrak{h} + \sum_{\alpha \in \Phi_j^+} (L_\alpha + L_{-\alpha})$, which is equivalent to $\mathfrak{a} = \mathfrak{h} + \sum_{\alpha \in P} L_\alpha$ with $P = \Phi_j^+ \cup -\Phi_j^+ = \Phi_J$. Then we have $P = -P$, and by proposition 2.2.6, we conclude that $\mathfrak{a}$ is reductive in $L$.

Proposition 2.2.8. Let $\mathfrak{t}$ be a subalgebra of $L$. Then $\mathfrak{t}$ is reductive in $L$ if and only if $[\mathfrak{t}, \mathfrak{t}]$ is semisimple.

Proof. See proposition 5, section 4, paragraph 6 from [Bou1].

Lemma 2.2.9. The subalgebra $a' = \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle$ is semisimple.

Proof. By lemma 2.2.4, $a'$ is the derived algebra of $a$. And by lemma 2.2.7, we know that $a$ is reductive in $L$. Applying then proposition 2.2.8, we obtain the result.

2.2.2 The module $V_0(I_J)$

We keep the notation introduced in the previous part, for $J, \Phi_J, I_J, \mathfrak{a}, \mathfrak{a}'$ and $P_J$.

Lemma 2.2.10. Let $I$ be an ideal of a subalgebra $X$ of $L$, and $V$ an $L$-module. Then the set $V_0(I) = \{v \in V \mid xv = 0 \ \forall x \in I\}$ is $X$-invariant.
Suppose for a contradiction that and Let where.

Then we can apply Weyl’s theorem of complete reducibility (theorem 1.2.2), and write Lemma 2.2.11.

interested in for every Let Proof. We apply lemma 2.2.10 to implying that is a -module, so it is also an -module, and consequently an -module.

Theorem 2.2.12. The set is an irreducible -module of highest weight .

Proof. Let be a maximal vector of weight . This means that is non zero, and for every , so we have . Since is semisimple by lemma 2.2.9, we can apply Weyl’s theorem of complete reducibility (theorem 1.2.2), and write as the direct sum of the module generated by and another -module . Suppose for a contradiction that . Then there is an which is another maximal vector for , with the Cartan subalgebra of .

Then we have

1) for so

2) for so

So for every , we get .

If is not a weight vector for , we see as a vector in the -module . By proposition 1.2.7, is the sum of its weight spaces, since is semisimple. Then we can write with a weight vector of weight for each . Then we obtain . But each is in and since we can chose the to be distinct, the spaces are also distinct. This means that for every and every in the sum, since the sum of the weight spaces is direct. Then if is not itself a weight vector, we can find some which is a weight vector for and which is a maximal vector for . Let write for this maximal vector.

In conclusion, must be a scalar multiple of , so is an -module, and consequently . This implies that is an irreducible -module, of highest weight .

2.2.3 Parabolic subalgebras, Levi subalgebras and nilpotent radical

Definition 2.2.13. A subalgebra of a Lie algebra is a parabolic subalgebra of if it contains a Borel subalgebra of .
Since a parabolic subalgebra \( p \) contains a Borel subalgebra of \( L \), it also contains a Cartan subalgebra \( h \). By the corresponding decomposition of \( L \) into the root spaces \( L_\alpha \), we can express \( p \) as \( h + \sum_{\alpha \in P} L_\alpha \), for \( h \) the Cartan subalgebra contained in \( p \) and \( P \) a subset of the associated root system \( \Phi \).

**Proposition 2.2.14.** A subalgebra \( p = h + \sum_{\alpha \in P} L_\alpha \) of \( L \), with \( h \) a Cartan subalgebra, and \( p \subset \Phi \) is parabolic if and only if \( P \cup -P = \Phi \).

**Proof.** See proposition 11, section 4, paragraph 3, chapter VIII from [Bou7-9]. \( \square \)

**Lemma 2.2.15.** The subalgebra \( P_J \) defined in this chapter is a parabolic subalgebra.

**Proof.** We defined \( P_J \) as \( \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle + h + \langle e_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_J \rangle \), so we can write

\[
P_J = h + \sum_{\alpha \in \Phi_J^+} (L_\alpha + L_{-\alpha}) + \sum_{\alpha \in \Phi^+ \setminus \Phi_J} L_\alpha
\]

\[
= h + \sum_{\alpha \in P} L_\alpha, \text{ with } P = \Phi_J^+ \cup (-\Phi_J^+) \cup \Phi^+ \setminus \Phi_J
\]

And we have \( P \cup (-P) = \Phi \), meaning that \( P_J \) is a parabolic subalgebra. \( \square \)

**Definition 2.2.16.** The intersection of the kernels of the simple finite-dimensional representations of \( L \) is the nilpotent radical of \( L \).

**Proposition 2.2.17.** Let \( p = h + \sum_{\alpha \in P} L_\alpha \) be a parabolic subalgebra of \( L \). The nilpotent radical of \( p \) is \( \sum_{\alpha \in Q} L_\alpha \), with \( Q = \{ \alpha \in P \mid -\alpha \not\in P \} \).

**Proof.** See proposition 13 section 4, paragraph 3, chapter VIII from [Bou7-9]. \( \square \)

**Lemma 2.2.18.** The ideal \( I_J \) is the nilpotent radical of the parabolic subalgebra \( P_J \) of \( L \).

**Proof.** We have seen before that \( P_J = h + \sum_{\alpha \in P} L_\alpha \), with \( P = \Phi_J^+ \cup (-\Phi_J^+) \cup \Phi^+ \setminus \Phi_J \). So by proposition 2.2.17, the nilpotent radical of \( P_J \) is equal to \( \sum_{\alpha \in Q} L_\alpha \), with \( Q = \{ \alpha \in P \mid -\alpha \not\in P \} \). In this case, it is clear that \( Q = \Phi^+ \setminus \Phi_J \), which implies that the nilpotent radical of \( P_J \) is the ideal generated by \( \{ e_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_J \} \), which is exactly \( I_J \) by lemma 2.2.1. \( \square \)

**Definition 2.2.19.** Let \( p \) be a parabolic subalgebra of a semisimple Lie algebra \( L \), and let \( n \) be the nilpotent radical of \( p \). A subalgebra \( l \) of \( L \) is a Levi subalgebra of \( p \) if \( p = l \oplus n \) and if \( l \) is reductive in \( L \).

**Theorem 2.2.20.** Let \( L \) be a semisimple Lie algebra, \( n \) a subalgebra of \( L \) consisting of nilpotent elements and \( s \) a semisimple Lie subalgebra of \( L \) which normalizes \( n \). Then there exists a parabolic subalgebra \( p \) of \( L \) such that

1) \( n \) is contained in the nilpotent radical of \( p \),

2) the normalizer of \( n \) in \( L \) is contained in \( p \), in particular \( s \subset p \),

3) there is a Levi subalgebra of \( p \) which contains \( s \).

**Proof.** See theorem 29.8.3 of [Tau] and its corollary 29.8.4. The proof of corollary 29.8.4 implies that the parabolic subalgebra is the one given by theorem 29.8.3, so we can have the three conditions simultaneously. \( \square \)
Let $L_1 \subset L_2$ be semisimple Lie algebras, with Cartan subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$, root systems $\Phi_1$ and $\Phi_2$ and $\Delta_1$ a base of $\Phi_1$, $\Delta_2$ a base of $\Phi_2$.

Let $J_1 \subset \Delta_1$ and let $I_{J_1} = \langle e_{\alpha} \mid \alpha \in \Phi_1^+ \backslash \Phi_{J_1} \rangle$, $a_1 = \langle e_{\alpha}, f_{\alpha} \mid \alpha \in J_1 \rangle + \mathfrak{h}_1$, and $P_{J_1} = a_1 \oplus I_{J_1}$ the corresponding parabolic subalgebra of $L_1$. Let $a'_1$ be the derived algebra of $a_1$.

**Proposition 2.2.21.** With these notations, there exists a parabolic subalgebra $\mathfrak{p}$ of $L_2$ such that $P_{J_1} \subset \mathfrak{p}$, $I_{J_1}$ is contained in the nilpotent radical of $\mathfrak{p}$ and $a'_1$ is contained in a Levi subalgebra of $\mathfrak{p}$.

**Proof.** We will apply theorem 2.2.20 to the subalgebra $I_{J_1}$ of $L_2$, with $\mathfrak{s} = a'_1$, which is semisimple by lemma 2.2.9. Since $I_{J_1}$ is an ideal of $P_{J_1}$ consisting of nilpotent elements by lemma 2.2.3, it is in particular a subalgebra of $L_2$. And $a'_1$ normalizes $I_{J_1}$ because $I_{J_1}$ is an ideal of $P_{J_1}$ and $a'_1$ is a subalgebra of $P_{J_1}$. Then by the theorem we know that there exists a parabolic subalgebra $\mathfrak{p}$ of $L_2$ such that $I_{J_1}$ is contained in the nilpotent radical of $\mathfrak{p}$ and there exists a Levi subalgebra of $\mathfrak{p}$ which contains $a_1$. Since $I_{J_1}$ is an ideal of $P_{J_1}$, we have $P_{J_1} \subset N_{L_2}(I_{J_1}) \subset N_{L_2}(P_{J_1})$, so by the second assumption we obtain $P_{J_1} \subset \mathfrak{p}$. □

Let $L_1 \subset L_2$ be two semisimple Lie algebras. Let $P_{J_1}, a_1, I_{J_1}$ be as defined before. Let $\mathfrak{p}$ be a parabolic subalgebra of $L_2$ as in proposition 2.2.21, and $\mathfrak{n}$ be its nilpotent radical.

We know that we can choose a Cartan subalgebra $\mathfrak{h}_2$ of $L_2$ with associated root system $\Phi_2$ such that $\mathfrak{h}_2$ is contained in $\mathfrak{p}$. Then fix a base $\Delta_2$ of $\Phi_2$ such that the Borel subalgebra $\mathfrak{b}_2$ contained in $\mathfrak{p}$ is the standard one, with respect to $\mathfrak{h}_2$ and $\Delta_2$. Now we can express $\mathfrak{p}$ as $\mathfrak{h}_2 + \sum_{\alpha \in \mathfrak{p}} L_{\alpha}$, for $P \subset \Phi_2$. Its radical nilpotent is then given by $\mathfrak{n} = \sum_{\alpha \in \mathfrak{Q}} L_{\alpha}$ for $\mathfrak{Q} = \{ \alpha \in P \mid -\alpha \notin P \} \subset \Phi_2$.

**Theorem 2.2.22.** Let $V$ be an irreducible $L_2$-module, with highest weight $\lambda$.

1) If $V|_{L_1}$ is irreducible, then $V_0(\mathfrak{n}) = (V|_{L_1})_0(I_{J_1})$.

2) If $V|_{L_1}$ is the sum of two irreducible summands $W$ and $Z$, then one of the following holds :

i) $V_0(\mathfrak{n}) = W_0(I_{J_1})$,

ii) $V_0(\mathfrak{n}) = Z_0(I_{J_1})$,

iii) $V_0(\mathfrak{n}) = W_0(I_{J_1}) \oplus Z_0(I_{J_1})$.

**Remark.** This theorem will be used when we look at an embedding of semisimple Lie algebras $L_1 \subset L_2$ with the aim of determining when the restriction of an irreducible representation of $L_2$ has exactly two irreducible $L_1$-summands. After making some calculations for small rank cases, we will be able to make a proof by induction, by considering the action of an appropriate smaller rank Levi subalgebra.

**Proof.** Since $V_0(\mathfrak{n})$ is a $\mathfrak{p}$-module by lemma 2.2.11, it is also an $a'_1$-submodule, since $a'_1 \subset \mathfrak{p}$ by choice of $\mathfrak{p}$ as in proposition 2.2.21. This proposition also states that $I_{J_1} \subset \mathfrak{n}$, which implies that we have $V_0(\mathfrak{n}) \subset (V|_{L_1})_0(I_{J_1})$.

This means that when $V|_{L_1}$ is irreducible, we have by theorem 2.2.12 that $(V|_{L_1})_0(I_{J_1})$ is an irreducible $a'_1$-module. By irreducibility we can then conclude that $V_0(\mathfrak{n}) = (V|_{L_1})_0(I_{J_1})$. 
On the other hand, if $V|_{L_1}$ decomposes into irreducible summands as $W \oplus Z$, we have by theorem 2.2.12 that $W_0(I_{J_1})$ and $Z_0(I_{J_1})$ are irreducible $\mathfrak{a}_1'$-modules. Let $v^+$ be the maximal vector of $V$, of weight $\lambda$. Then $v^+$ lies in $V_0(n)$ since $e_\alpha v^+ = 0$ for every $\alpha \in \Phi_2$. We have $v^+ \in V|_{L_1} = W \oplus Z$, and we can assume that $v^+ \in W$, since $v^+$ is a maximal vector for $b_1$. Then we have $v^+ \in V_0(n) \cap W_0(I_{J_1})$, since $v^+ \in W_0(I_{J_1})$ by the same argument. This means that $\{0\} \neq (V_0(n) \cap W_0(I_{J_1})) \subset W_0(I_{J_1})$. By the irreducibility of $W_0(I_{J_1})$, we conclude that

$$V_0(n) \cap W_0(I_{J_1}) = W_0(I_{J_1}). \quad (2.1)$$

But we know that $V_0(n) \subset (V|_{L_1})_0(I_{J_1}) = (W \oplus Z)_0(I_{J_1})$. By definition, we have the inclusion

$$W_0(I_{J_1}) \oplus Z_0(I_{J_1}) \subset (W \oplus Z)_0(I_{J_1}).$$

Suppose that

$$W_0(I_{J_1}) \oplus Z_0(I_{J_1}) \subsetneq (W \oplus Z)_0(I_{J_1}).$$

This means that there exists $w \in W$, $z \in Z$ and $x \in I_{J_1}$ such that $x(w + z) = 0$ and $xw$ or $xz$ is not zero. Then we must have $xw$ and $xz$ both non zero, but $x(w + z) = xw + xz = 0$ implies that $xw = -xz$. But $xw \in W$ and $xz \in Z$, so we obtain $xw = -xz \in W \cap Z$. Since we assumed that the sum of $W$ and $Z$ is direct, we have $W \cap Z = \{0\}$, meaning that $xw = 0 = xz$, which is a contradiction. This shows that

$$W_0(I_{J_1}) \oplus Z_0(I_{J_1}) = (W \oplus Z)_0(I_{J_1}).$$

In conclusion, we have $W_0(I_{J_1}) \subset V_0(n) \subset (W_0(I_{J_1}) \oplus Z_0(I_{J_1}))$, with the first inclusion deduced from 2.1, and by irreducibility of $W_0(I_{J_1})$ and $Z_0(I_{J_1})$ we obtain

$$V_0(n) = W_0(I_{J_1}),$$

or

$$V_0(n) = W_0(I_{J_1}) \oplus Z_0(I_{J_1}).$$

\qed
Chapter 3

Representations of $G_2$ restricted to $A_2$

We will start from an embedding of $A_2$ in $G_2$, and try to understand for which irreducible representations of $G_2$ we obtain exactly two irreducible summands upon restriction to the subalgebra $A_2$. We will be able to make this directly, so we will not use the induction technique in this chapter.

3.1 Embedding of $A_2$ in $G_2$

We recall that a Lie algebra of type $G_2$ has two simple roots $\alpha_1$ and $\alpha_2$. Its root system consists of six long roots ($\alpha_2, 3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2$ and their negatives), and six short roots ($\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2$ and their negatives).

The set of long roots of the root system of $G_2$ forms a root system of type $A_2$. We then have a subalgebra of type $A_2$ generated by $e_{\alpha_2}, f_{\alpha_2}, e_{3\alpha_1+\alpha_2}, f_{3\alpha_1+\alpha_2}$. Let us denote by $\beta_1$ and $\beta_2$ the two simple roots of $A_2$. By our inclusion of $A_2$ in $G_2$, we have the equalities $\beta_1 = \alpha_2$ and $\beta_2 = 3\alpha_1 + \alpha_2$. In order to avoid confusion, we will denote by $\lambda_1, \lambda_2$ the corresponding fundamental dominant weights of $G_2$ with respect to the base $\{\alpha_1, \alpha_2\}$, and by $\mu_1, \mu_2$ the corresponding fundamental dominant weights of $A_2$ with respect to the base $\{\beta_1, \beta_2\}$.

3.2 Statement and proof

**Proposition 3.2.1.** An irreducible representation of $G_2$ restricted to a representation of $A_2$, for the inclusion of $A_2$ in $G_2$ given by the long roots forming a system of type $A_2$, is never the sum of exactly two irreducible summands. Moreover, this restriction does not stay irreducible, unless the representation is trivial.

The proof will be separated in three cases, depending on the chosen irreducible representation of $G_2$. For a representation $V_{G_2}$ of $G_2$ with highest weight $\lambda = a\lambda_1 + b\lambda_2$, we will distinguish three cases, depending on the values of $a$ and $b$. The trivial representation with $\lambda = 0$ is not included here, since we know that the restriction of $V_{G_2}(0)$ to $A_2$ is the trivial representation $V_{A_2}(0)$.
CHAPTER 3. REPRESENTATIONS OF $G_2$ RESTRICTED TO $A_2$

Case 1: $\lambda = a\lambda_1$ with $a \neq 0$

First, we will compute the restriction of $\lambda$ to $A_2$. We know that

$$\lambda|_{A_2} = \lambda, 1 > \mu_1 + \lambda, 2 > \mu_2.$$  

Here for $\lambda = a\lambda_1$ we have

$$<\lambda, 1 > = a<\lambda_1, \beta_1 >, <\lambda, 2 > = a<\beta_2 >$$

and

$$<\lambda, 2 > = <\lambda, 3\alpha_1 + \alpha_2 > = <\sigma_1(\lambda), \sigma_1(3\alpha_1 + \alpha_2) > = <\lambda - a\alpha_1, \alpha_2 > = a,$$

where the second equality uses the fact that the product $<\cdot, \cdot>$ is invariant under the action of the Weyl group, but linear only in the first component. So we obtain the highest weight of a first irreducible summand of $V_{G_2}|_{A_2}$, which is $\mu = a\mu_2$.

Now we are searching for a second highest weight, and we see that $\nu = (\lambda - a\alpha_1)|_{A_2}$ is a highest weight for $A_2$, since $e_{\alpha_2}$ and $e_{3\alpha_1 + \alpha_2}$ act trivially on $V_{\lambda - a\alpha_1}$, and the weight is given by $\nu = a\mu_1$.

It is then enough to compare the dimensions of $V_{A_2}(\mu)$, $V_{A_2}(\nu)$ and $V_{G_2}(\lambda)$, to know whether the representation $V_{G_2}(\lambda)$ restricted to $A_2$ is the sum of $V_{A_2}(\mu)$ and $V_{A_2}(\nu)$ or not.

Recall that we have

$$\begin{cases} 
\lambda = a\lambda_1 \\
\mu = a\mu_2 \\
\nu = a\mu_1 
\end{cases}$$

By table 1.2, we obtain

$$\begin{cases} 
\dim(V_{G_2}(\lambda)) = \frac{1}{120}(a + 1)(a + 2)(a + 3)(a + 4)(2a + 5) \\
\dim(V_{A_2}(\mu)) = \frac{1}{2}(a + 1)(a + 2) \\
\dim(V_{A_2}(\nu)) = \frac{1}{2}(a + 1)(a + 2) 
\end{cases}$$

This means that $V_{G_2}(\lambda)|_{A_2} = V_{A_2}(\mu) \oplus V_{A_2}(\nu)$ if and only if

$$\frac{1}{120}(a + 1)(a + 2)(a + 3)(a + 4)(2a + 5) = \frac{1}{2}(a + 1)(a + 2) + \frac{1}{2}(a + 1)(a + 2),$$

which is equivalent to

$$(a + 1)(a + 2)(2a^3 + 19a^2 + 59a - 60) = 0.$$  

We can check that $2a^3 + 19a^2 + 59a - 60 = 0$ has no positive integer solution, so there is no $a \in \mathbb{N}$ such that the equality of dimensions

$$\dim(V_{G_2}(\lambda)) = \dim(V_{A_2}(\mu)) + \dim(V_{A_2}(\nu))$$

holds, and we conclude that the representation $V_{G_2}(\lambda)$ of $G_2$ decomposes always into at least three irreducible summands when restricted to $A_2$. 
Case 2: $\lambda = b\lambda_2$ with $b \neq 0$

First, the restriction of $\lambda$ to $A_2$ gives the highest weight $\mu = b\mu_1 + b\mu_2$. We find that $\lambda - \alpha_1 - \alpha_2$ is a weight, and we check that it is a highest weight since $e_{\alpha_2}$ and $e_{3\alpha_1 + \alpha_2}$ act trivially on the corresponding weight space, since $\lambda - \alpha_1$ is not a weight by lemma 1.2.13 because $\sigma_1(\lambda - \alpha_1) = \lambda + \alpha_1 \neq \lambda$. It is the maximal weight of an irreducible summand of $V_{A_2}$. We compute the restriction of $\lambda - \alpha_1 - \alpha_2$ to $A_2$, and we obtain the highest weight $\nu = (b - 1)\mu_1 + b\mu_2$ of a new irreducible summand.

Again, we have a look at the dimensions, to see if $V_{G_2}(\lambda)|_{A_2} = V_{A_2}(\mu) \oplus V_{A_2}(\nu)$. We have here

$$\dim(V_{G_2}(\lambda)) = \frac{1}{120}(b + 1)(b + 2)(3b + 4)(3b + 5),$$
$$\dim(V_{A_2}(\mu)) = (b + 1)^3,$$
$$\dim(V_{A_2}(\nu)) = \frac{1}{2}b(b + 1)(2b + 1).$$

We have the equality $\dim(V_{G_2}(\lambda)) = \dim(V_{A_2}(\mu)) + \dim(V_{A_2}(\nu))$ if and only if

$$\frac{1}{120}(b + 1)(b + 2)(3b + 4)(3b + 5) = (b + 1)^3 + \frac{1}{2}b(b + 1)(2b + 1),$$

which is true only when

$$b(b + 1)(18b^3 + 117b^2 + 43b + 2) = 0.$$ 

Since this equation has no positive integer solution, we conclude that $V_{G_2}(\lambda)$ does not decompose into two irreducible summands when restricted to $A_2$.

Case 3: $\lambda = a\lambda_1 + b\lambda_2$ with $ab \neq 0$

As usual, we start by computing the restriction of $\lambda$ to a highest weight $\mu$ of $A_2$. We have $\mu = \langle \lambda, \beta_1 \rangle \mu_1 + \langle \lambda, \beta_2 \rangle \mu_2 = b\mu_1 + (a + b)\mu_2$. We remark that the restriction of $\lambda - \alpha_1$ to $A_2$ gives the highest weight $\nu = (b + 1)\mu_1 + (a + b - 1)\mu_2$ of an irreducible summand, since $e_{\alpha_2}$ and $e_{3\alpha_1 + \alpha_2}$ can only act trivially on $V_{\lambda - \alpha_1}$.

When $a$ is greater than 1, we see by the same idea that $\lambda - 2\alpha_1$ restricted to $A_2$ is the highest weight of a third irreducible summand of the restriction, namely $\gamma = (b + 2)\mu_1 + (a + b - 2)\mu_2$. In this case, we have obtained at least three irreducible summands $V_{A_2}(\mu)$, $V_{A_2}(\nu)$ and $V_{A_2}(\gamma)$ of the restriction of $V_{G_2}$ to $A_2$.

If $a = 1$, we have found two highest weights, which allow us to compute the dimensions, in order to see if there is a third summand or not. So here $\lambda = \lambda_1 + b\lambda_2$ and we have the highest weights $\mu = b\mu_1 + (b + 1)\mu_2$ and $\nu = (b + 1)\mu_1 + b\mu_2$. We use table 1.2, and we get

$$\dim(V_{G_2}(\lambda)) = \frac{1}{30}(b + 1)(b + 2)(3b + 5)(3b + 7),$$
$$\dim(V_{A_2}(\mu)) = \frac{1}{2}(b + 1)(b + 2)(2b + 3),$$
$$\dim(V_{A_2}(\nu)) = \frac{1}{2}(b + 1)(b + 2)(2b + 3).$$

The representation $V_{G_2}(\lambda)$ decomposes in exactly two summands when restricted to $A_2$ if and only if

$$\frac{1}{30}(b + 1)(b + 2)(3b + 5)(3b + 7) = \frac{1}{2}(b + 1)(b + 2)(2b + 3) + \frac{1}{2}(b + 1)(b + 2)(2b + 3),$$

and equivalently

$$(b + 1)(b + 2)(9b^3 + 63b^2 + 83b + 15) = 0.$$ 

We see then that this equation has no positive integer solution, so we never have this equality of dimensions, and $V_{G_2}(\lambda)|_{A_2} \neq V_{A_2}(\mu) \oplus V_{A_2}(\nu)$.
Chapter 4

Representations of $D_{n+1}$ restricted to $B_n$

In this chapter, we will try to understand for which representations of $D_{n+1}$ we have a restriction to $B_n$ which decomposes in exactly two irreducible summands. We will use a chosen embedding, and work with it throughout the chapter. First, we look at the small cases where $n = 2$ and $n = 3$, and then we will use the induction step explained in section 2.2 to obtain the result for any $n \geq 2$.

4.1 Embedding of $B_n$ in $D_{n+1}$

Let $h_1$ be a Cartan subalgebra of $B_n$, let $\beta_1, \ldots, \beta_n$ be a base of the corresponding root system, and let $b_1$ be the standard corresponding Borel subalgebra. Similarly, let $h_2$ be a Cartan subalgebra of $D_{n+1}$, $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ a choice of simple roots and $b_2$ the standard corresponding Borel subalgebra.

We take the embedding of $B_n$ in $D_{n+1}$ given as follows: we take the subalgebra of type $B_n$ generated by $e_{\beta_1}, \ldots, e_{\beta_{n-1}}, e_{\beta_n} + e_{\alpha_{n+1}}; f_{\beta_1}, \ldots, f_{\beta_{n-1}}, f_{\beta_n} + f_{\alpha_{n+1}}$. This means that we identify $e_{\beta_i}$ with $e_{\alpha_i}$ for $i = 1, \ldots, n-1$ and $e_{\beta_n} = e_{\alpha_n} + e_{\alpha_{n+1}}$, and the same for $f_{\beta_i}$. In fact, for $n \neq 3$, we can observe that there is a unique embedding of $B_n$ into $D_{n+1}$, up to conjugation in $D_{n+1}$.

We can choose our Cartan and Borel subalgebras in a way such that $h_1 \subset h_2$ and $b_1 \subset b_2$.

From this, we obtain

$$h_{\beta_i} = [e_{\beta_i}, f_{\beta_i}] = [e_{\alpha_i}, f_{\alpha_i}] = h_{\alpha_i}, \text{ for } i = 1, \ldots, n-1$$

and

$$h_{\beta_n} = [e_{\beta_n}, f_{\beta_n}] = [e_{\alpha_n} + e_{\alpha_{n+1}}, f_{\alpha_n} + f_{\alpha_{n+1}}]$$

$$= [e_{\alpha_n}, f_{\alpha_n}] + [e_{\alpha_n}, f_{\alpha_{n+1}}] + [e_{\alpha_{n+1}}, f_{\alpha_n}] + [e_{\alpha_{n+1}}, f_{\alpha_{n+1}}]$$

$$= h_{\alpha_n} + h_{\alpha_{n+1}}.$$
4.2 Statement

**Proposition 4.2.1.** An irreducible representation $V_{D_{n+1}}(\lambda)$ of $D_{n+1}$ with highest weight $\lambda = \sum_{i=1}^{n+1} a_i \lambda_i$ restricted to a representation of $B_n$ is the sum of at most two irreducible summands if and only if $\lambda$ appears in the following table:

<table>
<thead>
<tr>
<th>Highest weight</th>
<th>Highest weights of the irreducible summands</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b\lambda_n$</td>
<td>$b\mu_n$</td>
</tr>
<tr>
<td>$b\lambda_{n+1}$</td>
<td>$b\mu_n$</td>
</tr>
<tr>
<td>$\lambda_1 + b\lambda_n$</td>
<td>$\mu_1 + b\mu_n / b\mu_n$</td>
</tr>
<tr>
<td>$\lambda_1 + b\lambda_{n+1}$</td>
<td>$\mu_1 + b\mu_n / b\mu_n$</td>
</tr>
<tr>
<td>$\lambda_i + b\lambda_n$ for $1 &lt; i &lt; n$</td>
<td>$\mu_i + b\mu_n / \mu_{i-1} + b\mu_n$</td>
</tr>
<tr>
<td>$\lambda_i + b\lambda_{n+1}$ for $1 &lt; i &lt; n$</td>
<td>$\mu_i + b\mu_n / \mu_{i-1} + b\mu_n$</td>
</tr>
<tr>
<td>$b\lambda_n + \lambda_{n+1}$, $b \neq 0$</td>
<td>$(b+1)\mu_n / \mu_{n-1} + (b-1)\mu_n$</td>
</tr>
<tr>
<td>$\lambda_n + b\lambda_{n+1}$, $b \neq 0$</td>
<td>$(b+1)\mu_n / \mu_{n-1} + (b-1)\mu_n$</td>
</tr>
</tbody>
</table>

Table 4.1: Irreducible $D_{n+1}$-modules decomposing into at most two irreducible summands when restricted to $B_n$.

The rest of this chapter leads to the proof of this statement, by induction on $n$. We will first handle the case $n = 2$. Then applying theorem 2.2.22, we will turn to the case $n = 3$ but only looking at those cases where the highest weight $\lambda$ for $V_{D_4}(\lambda)$ has a restriction to the $D_3$ Levi-subalgebra which is a weight found in the first case. This will prove the proposition for the cases $n = 2$ and $n = 3$, and we will then be able to conclude by showing the induction step, again with the help of theorem 2.2.22.

We will often need to compute the multiplicity of a weight in a representation, for which we will always use Freudenthal’s formula, presented in theorem 1.3.2, we will not mention it each time.

4.3 Representations of $D_3$ restricted to $B_2$

4.3.1 Statement

**Proposition 4.3.1.** An irreducible representation $V_{D_3}(\lambda)$ of $D_3$ with highest weight $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3$ restricted to the simple subalgebra of type $B_2$ is the sum of at most two irreducible summands if and only if $\lambda$ appears in the following table:

We separate the proof in different cases, depending on the highest weight $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3$, we make different cases depending whether two, one or none of the parameters $a, b, c$ are zero.

The trivial representation of $D_3$ will clearly stay trivial when restricted to $B_2$, so $V_{D_3}(0)|_{B_2} = V_{B_2}(0)$. Now we assume that at least one of the parameters $a, b, c$ is non zero, and we look at each case. We will also use the symmetry of $D_3$ to reduce the cases we need to work on.
Case 1: \( \lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 \) with \( abc \neq 0 \)

The restriction of \( \lambda \) to \( B_2 \) gives a weight \( \mu = a\mu_1 + (b+c)\mu_2 \). Since \( b \) and \( c \) are non-zero, we see that \( \lambda - \alpha_2 \) and \( \lambda - \alpha_3 \) are two weights of \( V_{D_3}(\lambda) \), both of them restricting to \( B_2 \) as \( \mu - \beta_2 \), which is dominant for \( B_2 \). But the weight \( \mu - \beta_2 \) appears with multiplicity only 1 in \( V_{B_2}(\mu) \), so there is a second irreducible summand having \( \mu - \beta_2 \) as a weight. By lemma 2.1.3, we observe that this irreducible summand is in fact \( V_{B_2}(\mu - \beta_2) = V_{B_2}(\nu) \).

Similarly, the two weights \( \lambda - \alpha_1 - \alpha_2 \) and \( \lambda - \alpha_1 - \alpha_3 \) are both of multiplicity 2 in \( V_{D_3}(\lambda) \), and correspond to \( \mu - \beta_1 - \beta_2 \). And we obtain that \( \mu - \beta_1 - \beta_2 \) is of multiplicity 2 in \( V_{B_2}(\mu) \) and 1 in \( V_{B_2}(\nu) \). Consequently, the weight \( \mu - \beta_1 - \beta_2 \) must appear in a third irreducible summand, so when \( abc \neq 0 \), the representation \( V_{D_3}(a\lambda_1 + b\lambda_2 + c\lambda_3) \) decomposes into at least three irreducible summands when restricted to \( B_2 \).

Case 2: \( \lambda = a\lambda_1 + b\lambda_2 \) with \( ab \neq 0 \)

We compute the restriction of \( \lambda \) and we obtain the highest weight of a first irreducible summand, \( \mu = a\mu_1 + b\mu_2 \). Then we see that \( \lambda - \alpha_1 - \alpha_2 \) and \( \lambda - \alpha_1 - \alpha_3 \) are weights restricting as \( \mu - \beta_1 - \beta_2 \). By Freudenthal’s formula, the multiplicity of \( \lambda - \alpha_1 - \alpha_2 \) in \( V_{D_3}(\lambda) \) is 2 and the multiplicity of \( \lambda - \alpha_1 - \alpha_3 \) is 1. On the other hand, the weight \( \mu - \beta_1 - \beta_2 \) appears with multiplicity 2 in \( V_{B_2}(\mu) \), which means that there is a second irreducible summand with highest weight \( \mu - \beta_1 - \beta_2 \), by lemma 2.1.3, which is \( V_{B_2}(\mu - \beta_1 - \beta_2) = V_{B_2}((a-1)\mu_1 + b\mu_2) = V_{B_2}(\nu) \).

Now if \( a \) and \( b \) are both greater than 1, we have the weights \( \lambda - 2\alpha_1 - \alpha_2 - \alpha_3 \), \( \lambda - 2\alpha_1 - 2\alpha_2 \) and \( \lambda - 2\alpha_1 - 2\alpha_3 \), all of them restricting to \( B_2 \) as \( \mu - 2\beta_1 - 2\beta_2 \). We compute the multiplicities and we find that the three weights appear in \( V_{D_3}(\lambda) \) with multiplicity equal to 3, 3 and 1, respectively. But in \( V_{B_2}(\mu) \), the weight \( \mu - 2\beta_1 - 2\beta_2 \) has multiplicity 4, and it has multiplicity 2 in \( V_{B_2}(\nu) \), meaning that \( \mu - 2\beta_1 - 2\beta_2 \) is also the weight of a third irreducible summand.

Now if \( a \) or \( b \) is equal to 1, as we found two irreducible summands, we can use the dimensions in order to see if there is a third irreducible summand or not.

<table>
<thead>
<tr>
<th>Highest weight</th>
<th>Highest weights of the irreducible summands</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b\lambda_2 )</td>
<td>( b\mu_2 )</td>
</tr>
<tr>
<td>( b\lambda_3 )</td>
<td>( b\mu_2 )</td>
</tr>
<tr>
<td>( \lambda_1 + b\lambda_2 )</td>
<td>( \mu_1 + b\mu_2 / b\mu_2 )</td>
</tr>
<tr>
<td>( \lambda_1 + b\lambda_3 )</td>
<td>( \mu_1 + b\mu_2 / b\mu_2 )</td>
</tr>
<tr>
<td>( b\lambda_2 + \lambda_3, b \neq 0 )</td>
<td>( (b + 1)\mu_2 / \mu_1 + (b - 1)\mu_2 )</td>
</tr>
<tr>
<td>( \lambda_2 + b\lambda_3, b \neq 0 )</td>
<td>( (b + 1)\mu_2 / \mu_1 + (b - 1)\mu_2 )</td>
</tr>
</tbody>
</table>
For $a = 1$, we obtain
\[
\begin{align*}
\dim(V_{D_3}(\lambda_1 + b\lambda_2)) &= \frac{1}{4} (b + 1)(b + 3)(b + 4), \\
\dim(V_{B_2}(\mu_1 + b\mu_2)) &= \frac{1}{3} (b + 1)(b + 3)(b + 5), \\
\dim(V_{B_2}(b\mu_2)) &= \frac{1}{6} (b + 1)(b + 2)(b + 3)
\end{align*}
\]
and by developing each term, we see that we for $b > 0$
\[
\dim(V_{D_3}(\lambda_1 + b\lambda_2)) = \dim(V_{B_2}(\mu_1 + b\mu_2)) + \dim(V_{B_2}(b\mu_2)),
\]
so the case $a = 1$ appears as an example in table 4.2, since then
\[
V_{D_3}(\lambda_1 + b\lambda_2)|_{B_2} = V_{B_2}(\mu_1 + b\mu_2) \oplus V_{B_2}(b\mu_2).
\]

When $b = 1$, we have
\[
\begin{align*}
\dim(V_{D_3}(a\lambda_1 + \lambda_2)) &= \frac{1}{3} (a + 1)(a + 3)(a + 4), \\
\dim(V_{B_2}(a\mu_1 + \mu_2)) &= \frac{2}{3} (a + 1)(a + 2) \\
\dim(V_{B_2}((a - 1)\mu_1 + \mu_2)) &= \frac{2}{3} a(a + 1)(a + 2)
\end{align*}
\]
This means that
\[
\dim(V_{D_3}(a\lambda_1 + \lambda_2)) = \dim(V_{B_2}(a\mu_1 + \mu_2)) + \dim(V_{B_2}((a - 1)\mu_1 + \mu_2))
\]
if and only if
\[
\frac{1}{6} (a + 1)(a + 2)(a^2 - a) = 0,
\]
which means that the previous equality holds exactly when $a = 0$ or $a = 1$. We were in the case $a \neq 0$ by hypothesis, so we conclude that
\[
V_{D_3}(\lambda_1 + b\lambda_2)|_{B_2} = V_{B_2}(\mu_1 + b\mu_2) \oplus V_{B_2}(b\mu_2),
\]
and if $a \neq 1$, the restriction of $V_{D_3}(a\lambda_1 + b\lambda_2)$ has at least three irreducible summands.

**Case 3:** $\lambda = b\lambda_2 + c\lambda_3$ with $bc \neq 0$

The restriction of $\lambda$ is given by $\mu = (b + c)\mu_2$. We assumed $b$ and $c$ non zero, so $\lambda - \alpha_2$ and $\lambda - \alpha_3$ are weights, which both restrict as $\nu = \mu - \beta_2 = \mu_1 + (b + c - 2)\mu_2$. But by Freudenthal’s formula, we compute the multiplicity of $\nu$ in $V_{B_2}(\mu)$, which is equal to 1. Since the corresponding weights have a multiplicity of minimum 1 each in $V_{D_3}(\lambda)$, the weight $\nu$ must appear in a new irreducible summand. So we know that there is a second irreducible summand, and by lemma 2.1.3, it is given by $V_{B_2}(\nu)$.

When $b$ and $c$ are both greater than 1, we know that $\lambda - 2\alpha_2$, $\lambda - 2\alpha_3$ and $\lambda - \alpha_2 - \alpha_3$ are three weights, all of them restricting to $B_2$ as $\gamma = \mu - 2\beta_2 = 2\mu_1 + (b + c - 4)\mu_2$. But the multiplicity of $\gamma$ is 1 in $V_{B_2}(\mu)$ and 1 in $V_{B_2}(\nu)$, so $\gamma$ gives a third irreducible summand.

We just need to look at the case where $b$ and $c$ are not both greater than 1, i.e., $c = 1$ (or $b = 1$, by symmetry). Here we consider the dimensions.

We have here
\[
\begin{align*}
\dim(V_{D_3}(b\lambda_2 + \lambda_3)) &= \frac{1}{2} (b + 1)(b + 2)(b + 3), \\
\dim(V_{B_2}((b + 1)\mu_2)) &= \frac{1}{6} (b + 2)(b + 3)(b + 4), \\
\dim(V_{B_2}(\mu_1 + (b - 1)\mu_2)) &= \frac{3}{2} (b + 2)(b + 4)
\end{align*}
\]
Since the sum of the dimensions of the two representations of $B_2$ is always equal to the dimension of the representation of $D_3$ here, we have
\[
V_{D_3}(b\lambda_2 + \lambda_3)|_{B_2} = V_{B_2}((b + 1)\mu_2) \oplus V_{B_2}(\mu_1 + (b - 1)\mu_2).
\]
Case 4: $\lambda = a\lambda_1$ with $a \neq 0$

The restriction of the weight $\lambda = a\lambda_1$ to $B_2$ gives the highest weight $\mu = a\mu_1$. Then we see that $\lambda - \alpha_1 - \alpha_2$ and $\lambda - \alpha_1 - \alpha_3$ are weights of $V_{D_3}(\lambda)$, both corresponding to $\mu - \beta_1 - \beta_2$ when restricted to $B_2$. Now we use Freudenthal's formula to compute the multiplicity of $\mu - \beta_1 - \beta_2$ in $V(\mu)$, and we obtain $m(\mu - \beta_1 - \beta_2) = 1$ in $V_{B_2}(\mu)$. Since there are two weights of $V_{D_3}$ corresponding to $\mu - \beta_1 - \beta_2$, this implies that $\mu - \beta_1 - \beta_2$ must appear in another irreducible summand of the restriction of $V_{D_3}(\lambda)$. We get here a second irreducible summand, that has highest weight $\nu = (a - 1)\mu_1$ by lemma 2.1.3.

As usual, we conclude by the dimensions, which are here equal to

$$
\begin{align*}
\dim(V_{D_3}(a\lambda_1)) &= \frac{1}{12}(a + 1)(a + 2)(a + 3)(a + 3) \\
\dim(V_{B_2}(a\mu_1)) &= \frac{1}{6}(a + 1)(a + 2)(2a + 3) \\
\dim(V_{B_2}((a - 1)\mu_1)) &= \frac{1}{6}a(a + 1)(2a + 1)
\end{align*}
$$

We have

$$
\dim(V_{D_3}(a\lambda_1)) = \dim(V_{B_2}(a\mu_1)) + \dim(V_{B_2}((a - 1)\mu_1))
$$

if and only if

$$
a^2(a + 1)(a - 1) = 0.
$$

This means that for $a = 1$, we have

$$
V_{D_3}(\lambda_1)|_{B_2} = V_{B_2}(\mu_1) \oplus V_{B_2}(0),
$$

and if $a$ is not equal to 1, then there are at least three irreducible summands in the decomposition.

Case 5: $\lambda = b\lambda_2$ with $b \neq 0$

Here we have $\lambda = b\lambda_2$, and the restriction $\mu$ of $\lambda$ is simply given by $\mu = b\mu_2$. We compare now the dimensions of the representation and its restriction, and we have

$$
\dim(V_{D_3}(b\lambda_2)) = \frac{1}{6}(b + 1)(b + 2)(b + 3),
$$

and

$$
\dim(V_{B_2}(b\mu_2)) = \frac{1}{6}(b + 1)(b + 2)(b + 3).
$$

We see that the dimensions are always the same, so the restriction of $V_{D_3}(\lambda)$ is equal to $V_{B_2}(\mu)$, which is also irreducible.

This completes the proof of proposition 4.3.1.

4.4 Representations of $D_4$ restricted to $B_3$

We study now the case of $B_3 \subset D_4$, before giving a general proof of proposition 4.2.1 by induction.
CHAPTER 4. REPRESENTATIONS OF $D_{N+1}$ RESTRICTED TO $B_N$

Highest weight | Highest weights of the irreducible summands |
---|---|
$b\lambda_3$ | $b\mu_3$ |
$b\lambda_4$ | $b\mu_3$ |
$\lambda_1 + b\lambda_3$ | $\mu_1 + b\mu_3 / b\mu_3$ |
$\lambda_1 + b\lambda_4$ | $\mu_1 + b\mu_3 / b\mu_3$ |
$\lambda_2 + b\lambda_3$ | $\mu_2 + b\mu_3 / \mu_1 + b\mu_3$ |
$\lambda_2 + b\lambda_4$ | $\mu_2 + b\mu_3 / \mu_1 + b\mu_3$ |
$b\lambda_3 + \lambda_4$, $b \neq 0$ | $(b+1)\mu_3 / \mu_2 + (b-1)\mu_3$ |
$\lambda_3 + b\lambda_4$, $b \neq 0$ | $(b+1)\mu_3 / \mu_2 + (b-1)\mu_3$ |

Table 4.3: Irreducible $D_4$-modules decomposing into at most two irreducible summands when restricted to $B_3$.

### 4.4.1 Statement

**Proposition 4.4.1.** An irreducible representation $V_{D_4}(\lambda)$ of $D_4$ with highest weight $\lambda = d\lambda_1 + a\lambda_2 + b\lambda_3 + c\lambda_4$ restricted to a representation of $B_3$ is the sum of at most two irreducible summands if and only if $\lambda$ appears in the following table:

Under the hypothesis of proposition 4.4.1, by the theorem 2.2.22 and by symmetry, we may assume that the restriction of $\lambda$ to the subalgebra $a' = \langle e_{\alpha_i}, f_{\alpha_i} \mid i \geq 2 \rangle$ is of the form $a\lambda_2 + b\lambda_3 + c\lambda_4$ with:

1) $(a, b, c) = (0, b, 0)$, $b \geq 0$, 
2) $(a, b, c) = (1, b, 0)$, $b \geq 0$, 
3) $(a, b, c) = (0, b, 1)$, $b \geq 1$.

Now for the proof of the proposition, we analyse what happens in each of these situations.

### 4.4.2 Proof

**Case 1:** $\lambda = d\lambda_1 + b\lambda_3$

The restriction of $\lambda$ to $B_3$ gives the weight $\mu = d\mu_1 + b\mu_3$.

When $d = 0$, we can see by the dimension that the restriction of $V_{D_4}(\lambda)$ is exactly $V_{B_3}(\mu)$, so the restriction of the representation stays irreducible. Indeed, we have

$$\dim(V_{D_4}(b\lambda_3)) = \frac{1}{360} (b+1)(b+2)(b+3)^2(b+4)(b+5) = \dim(V_{B_3}(b\mu_3)).$$

If $d$ is not equal to zero, then $\lambda - \alpha_1 - \alpha_2 - \alpha_3$ is a weight, and so is $\lambda - \alpha_1 - \alpha_2 - \alpha_4$. Their restriction is equal to $\nu = \mu - \beta_1 - \beta_2 - \beta_3 = (d-1)\mu_1 + b\mu_3$. With Freudenthal’s formula, we obtain the multiplicities, which are 3 for $\lambda - \alpha_1 - \alpha_2 - \alpha_3$ in $V_{D_4}(\lambda)$, 1 for $\lambda - \alpha_1 - \alpha_2 - \alpha_4$ in $V_{D_4}(\lambda)$ and 3 for $\mu - \beta_1 - \beta_2 - \beta_3$ in $V_{B_3}(\mu)$. This means that
Case 2: $\lambda = d\lambda_1 + \lambda_2 + b\lambda_3$

First we see that the restriction of $\lambda$ to $B_3$ gives the weight $\mu = d\mu_1 + \mu_2 + b\mu_3$.

Then we obtain that $\lambda - \alpha_2 - \alpha_3$ and $\lambda - \alpha_2 - \alpha_4$ are two weights restricting to $B_3$ as $\mu - \beta_2 - \beta_3$. The multiplicities obtained by Freudenthal’s formula are 2 for $\lambda - \alpha_2 - \alpha_3$ and 1 for $\lambda - \alpha_2 - \alpha_4$ in $V_{D_4}(\lambda)$, and only 2 for $\mu - \beta_2 - \beta_3$ in $V_{B_3}(\mu)$, which implies that there must be a second irreducible summand. Since $\mu - \beta_2 - \beta_3$ has to be a weight of this new summand, we use lemma 2.1.3 again and we see that this summand is $V_{B_3}(\mu - \beta_2 - \beta_3) = V_{B_3}(\nu) = V_{B_3}((d + 1)\mu_1 + b\mu_3)$.

If $d$ is non zero, we observe that $\lambda - \alpha_1 - \alpha_2 - \alpha_3$ and $\lambda - \alpha_1 - \alpha_2 - \alpha_4$ are weights of $V_{D_4}(\lambda)$ which both restrict as $\mu - \beta_1 - \beta_2 - \beta_3$. In $V_{D_4}(\lambda)$, the two weights appear with multiplicity 4 and 2, respectively. But the weight $\mu - \beta_1 - \beta_2 - \beta_3$ has multiplicity 4 in $V_{B_3}(\mu)$ and 1 in $V_{B_3}(\nu)$, and we see that $\mu - \beta_1 - \beta_2 - \beta_3$ is also a weight of a third irreducible summand.
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This means that when $d$ is non zero, we have at least three irreducible summands. For the case $d = 0$, we use the dimensions:

\[
\begin{aligned}
\dim(V_{D_4}(\lambda_2 + b\lambda_3)) &= \frac{1}{60}(b + 1)(b + 3)(b + 4)(b + 4)(b + 5)(b + 7) \\
\dim(V_{B_3}(\mu_2 + b\mu_3)) &= \frac{1}{120}(b + 1)(b + 3)(b + 4)(b + 5)(b + 6)(b + 7) \\
\dim(V_{B_3}(\mu_1 + b\mu_3)) &= \frac{1}{120}(b + 1)(b + 2)(b + 3)(b + 4)(b + 5)(b + 7)
\end{aligned}
\]

and we get that

\[
\dim(V_{D_4}(\lambda_2 + b\lambda_3)) = \dim(V_{B_3}(\mu_2 + b\mu_3)) + \dim(V_{B_3}(\mu_1 + b\mu_3))
\]

if and only if

\[
(b + 1)(b + 3)(b + 4)(b + 5)(b + 7)(\frac{b + 4}{60} - \frac{b + 6}{120} - \frac{b + 2}{120}) = 0,
\]

which is always true, and that means that

\[
V_{D_4}(\lambda_2 + b\lambda_3)|_{B_3} = V_{B_3}(\mu_2 + b\mu_3) \oplus V_{B_3}(\mu_1 + b\mu_3).
\]

**Case 3:** $\lambda = d\lambda_1 + b\lambda_3 + \lambda_4$

We can assume here that $b$ is non zero, since by symmetry the case $b = 0$ is just one of the possibilities of the first case of this proof.

As usual, we start by the restriction of $\lambda$, which is here equal to $\mu = d\mu_1 + (b + 1)\mu_3$. Then we see that $\lambda - \alpha_3$ and $\lambda - \alpha_4$ are weights corresponding to the restriction $\nu = \mu - \beta_3 = d\mu_1 + \mu_2 + (b - 1)\mu_3$. We use Freudenthal’s formula to compute the multiplicity of $\nu = \mu - \beta_3$ in $V_{B_3}(\mu)$ and we find that it is equal to 1. Since this weight corresponds to two distinct weights in $V_{D_4}(\lambda)$, we know that we have a new irreducible summand, namely $V_{B_3}(\nu)$ by lemma 2.1.3.

When $d$ is equal to zero, we use the dimensions and we obtain

\[
\begin{aligned}
\dim(V_{D_4}(b\lambda_3 + \lambda_4)) &= \frac{1}{90}(b + 1)(b + 2)(b + 3)(b + 4)(b + 5)(b + 6) \\
\dim(V_{B_3}((b + 1)\mu_3)) &= \frac{1}{360}(b + 2)(b + 3)(b + 4)(b + 5)(b + 6) \\
\dim(V_{B_3}(\mu_2 + (b - 1)\mu_3)) &= \frac{1}{120}b(b + 2)(b + 3)(b + 4)(b + 5)(b + 6)
\end{aligned}
\]

and the equality

\[
\dim(V_{D_4}(b\lambda_3 + \lambda_4)) = \dim(V_{B_3}((b + 1)\mu_3)) + \dim(V_{B_3}(\mu_2 + (b - 1)\mu_3))
\]

is true for any value of $b$, which means that

\[
V_{D_4}(b\lambda_3 + \lambda_4)|_{B_3} = V_{B_3}((b + 1)\mu_3) \oplus V_{B_3}(\mu_2 + (b - 1)\mu_3).
\]

Now if $d$ is non zero, we have that $\lambda - \alpha_1 - \alpha_2 - \alpha_3$ is a weight, with restriction $\gamma = \mu - \beta_1 - \beta_2 - \beta_3$, which is also the restriction of the weight $\lambda - \alpha_1 - \alpha_2 - \alpha_4$. The weight $\gamma$ has multiplicity 3 in $V_{B_3}(\mu)$ and 2 in $V_{B_3}(\nu)$, when the weights $\lambda - \alpha_1 - \alpha_2 - \alpha_3$ and $\lambda - \alpha_1 - \alpha_2 - \alpha_4$ appear with multiplicity 3 for each of them in $V_{D_4}(\lambda)$. We conclude then that for $d \neq 0$, there is a third irreducible summand.
4.5 Inductive step

Now that we have handled the cases \( n = 2 \) and \( n = 3 \), we will be able to complete the proof of proposition 4.2.1 by showing the inductive step. So we suppose that the proposition holds for \( B_{n-1} \) in \( D_n \) and we will show using theorem 2.2.22 that it has to be true for \( B_n \) in \( D_{n+1} \). In fact, theorem 2.2.22 allows us to just let the coefficient of \( \lambda_1 \) change, and we have the following possibilities for \( \lambda \) restricted to \( B_n \) in \( D_n + 1 \):

1) \((a_2, \ldots, a_{n+1}) = (0, \ldots, 0, b, 0), \ b \geq 0,\)
2) \((a_2, \ldots, a_{n+1}) = (0, \ldots, 0, 1, 0, \ldots, 0, b, 0), \ b \geq 0, \) with \( a_i = 1, \) and \( 2 \leq i < n,\)
4) \((a_2, \ldots, a_{n+1}) = (0, \ldots, 0, b, 1), \ b \geq 1,\)

We will need to compute the multiplicities of some weights, and for this purpose we will always use Freudenthal’s formula, lemma 1.3.5, lemma 1.3.6 and proposition 1.3.4, but the details of the calculations of the multiplicities will not be written here, for an easier understanding of the proof.

4.5.1 Proof

Case 1 : \( \lambda = d\lambda_1 + b\lambda_n \)

Restricting \( \lambda \) to \( B_n \) gives the weight \( \mu = d\mu_1 + b\mu_n \) of a first irreducible summand. If \( d = 0 \), lemma 1.3.9 shows that we have \( V_{D_{n+1}}(\lambda)|_{B_n} = V_{B_n}(\mu) \).

When \( d \) is not equal to zero, we see that \( \lambda - \sum_{i=1}^{n} \alpha_i \) and \( \lambda - \sum_{i=1}^{n-1} \alpha_i + \alpha_{n+1} \) are weights, which both restrict to \( B_n \) as \( \mu - \sum_{i=1}^{n} \beta_i = \nu \). We compute the multiplicities, and we obtain

<table>
<thead>
<tr>
<th>Case ( b = 0 )</th>
<th>Case ( b \geq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{V_{B_n}(\mu)}(\nu) )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{V_{D_{n+1}}(\lambda)}(\lambda - \sum_{i=1}^{n} \alpha_i ) )</td>
<td>1</td>
</tr>
<tr>
<td>( m_{V_{D_{n+1}}(\lambda)}(\lambda - \sum_{i=1}^{n-1} \alpha_i - \alpha_{n+1} ) )</td>
<td>1</td>
</tr>
<tr>
<td>Total multiplicity in ( V_{D_{n+1}}(\lambda) )</td>
<td>2</td>
</tr>
<tr>
<td>Difference between the multiplicities</td>
<td>1</td>
</tr>
</tbody>
</table>

So there must be a new irreducible summand, having \( \nu \) as a weight. By lemma 2.1.3, we deduce that this irreducible summand is exactly \( V_{B_n}(\nu) = V_{B_n}((d-1)\mu_1 + b\mu_n) \).

For \( d = 1 \), we use the dimensions, namely lemma 1.3.10, and we obtain

\( V_{D_{n+1}}(\lambda)|_{B_n} = V_{B_n}(\mu) \oplus V_{B_n}(\nu) \).

If \( d \) is greater than 1, we observe that \( \lambda - 2 \sum_{i=1}^{n} \alpha_i, \lambda - 2 \sum_{i=1}^{n-1} \alpha_i - 2\alpha_{n+1} \) and \( \lambda - 2 \sum_{i=1}^{n-1} \alpha_i - \alpha_n - \alpha_{n+1} \) are three weights, all of them restricting as \( \mu - 2 \sum_{i=1}^{n} \beta_i = \gamma \). Now we will use the multiplicities to show that this weight must appear in a third irreducible summand.

We summarize our results in the following table:
CHAPTER 4. REPRESENTATIONS OF $D_{N+1}$ RESTRICTED TO $B_{N}$

<table>
<thead>
<tr>
<th>Case $b = 0$</th>
<th>Case $b = 1$</th>
<th>Case $b \geq 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{V_{D_{n}}(\mu)}(\gamma)$</td>
<td>$n$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$m_{V_{D_{n}}(\nu)}(\gamma)$</td>
<td>$1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$m_{V_{D_{n+1}}(\lambda)}(\lambda - 2 \sum_{i=1}^{n} \alpha_i)$</td>
<td>$1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$m_{V_{D_{n+1}}(\lambda)}(\lambda - 2 \sum_{i=1}^{n-1} \alpha_i - 2\alpha_{n+1})$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$m_{V_{D_{n+1}}(\lambda)}(\lambda - 2 \sum_{i=1}^{n-1} \alpha_i - \alpha_n - \alpha_{n+1})$</td>
<td>$n$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Total multiplicity in $V_{D_{n+1}}(\lambda)$</td>
<td>$n + 2$</td>
<td>$\frac{n^2 + 3n + n}{2}$</td>
</tr>
<tr>
<td>Difference between the multiplicities</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Since the difference between the multiplicities in $V_{D_{n+1}}$ and in the restriction is always 1, we know that there is at least a third irreducible summand.

Hence we deduce that in case 1, we have the following results:

1) When $\lambda = b\lambda_n$, the restriction of $V_{D_{n+1}}(\lambda)$ to $B_n$ stays irreducible.

2) When $\lambda = \lambda_1 + b\lambda_n$, the restriction of $V_{D_{n+1}}(\lambda)$ to $B_n$ decomposes in the two irreducible summands $V_{B_n}(\mu_1 + b\mu_n)$ and $V_{B_n}(b\mu_n)$.

3) When $\lambda = b\lambda_1 + b\lambda_n$ with $d > 1$, there are at least three irreducible summands appearing in the restriction to $B_n$.

Case 2: $\lambda = d\lambda_1 + \lambda_i + b\lambda_n$, with $1 < i < n$

The restriction of $\lambda$ to $B_n$ gives the weight $\mu = d\mu_1 + \mu_i + b\mu_n$. Then we have the two weights $\lambda - \sum_{j=1}^{n} \alpha_j$ and $\lambda - \sum_{j=1}^{n-1} \alpha_j - \alpha_{n+1}$ which restrict to $B_n$ as $\nu = \mu - \sum_{j=1}^{n} \beta_i$.

Again, we compute the multiplicities, and we get

<table>
<thead>
<tr>
<th>Case $b = 0$</th>
<th>Case $b \geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{V_{B_n}(\mu)}(\nu)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$m_{V_{D_{n+1}}(\lambda)}(\lambda - \sum_{j=1}^{n} \alpha_j)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$m_{V_{D_{n+1}}(\lambda)}(\lambda - \sum_{j=1}^{n-1} \alpha_j - \alpha_{n+1})$</td>
<td>$1$</td>
</tr>
<tr>
<td>Total multiplicity in $V_{D_{n+1}}(\lambda)$</td>
<td>$2$</td>
</tr>
<tr>
<td>Difference between the multiplicities</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Then we see that $\mu - \sum_{j=1}^{n} \beta_n$ is the highest weight of a new irreducible summand, namely $V_{B_n}(\nu) = d\mu_1 + \mu_i - 1 + b\mu_n$ by lemma 2.1.3.

If $d = 0$, we use the dimensions to show that $V_{D_{n+1}}(\lambda) = V_{B_n}(\mu) \oplus V_{B_n}(\nu)$, which is a consequence of lemma 1.3.11.

And when $d \neq 0$, we have the two weights $\lambda - \sum_{j=1}^{n} \alpha_j$ and $\lambda - \sum_{j=1}^{n-1} \alpha_j - \alpha_{n+1}$, each of which restricts to $B_n$ as $\gamma = \mu - \sum_{j=1}^{n} \beta_j$. As usual, we compute the multiplicities and we obtain
Since the multiplicity of $\gamma$ in the restriction is strictly smaller than in $V_{D_{n+1}}(\lambda)$, we conclude that there is a third irreducible summand when $d \neq 0$.

Now in case 2, we found that:

1) When $\lambda = \lambda_i + b\lambda_n$, the restriction of $V_{D_{n+1}}(\lambda)$ to $B_n$ consists in the two irreducible summands $V_{B_n}(\mu_i + b\mu_n)$ and $V_{B_n}(\mu_{i-1} + b\mu_n)$.

2) When $\lambda = d\lambda_1 + \lambda_i + b\lambda_n$ with $d > 0$, there are at least three irreducible summands in the restriction of $V_{D_{n+1}}(\lambda)$ to $B_n$.

**Case 3 : $\lambda = d\lambda_1 + b\lambda_n + \lambda_{n+1}$, with $b \neq 0$**

As usual, we start by the restriction of $\lambda$ and we find the maximal weight of a first irreducible summand of $V_{D_{n+1}}(\lambda)|_{B_n}$. Here we have $\mu = d\mu_1 + (b + 1)\mu_n$. Then we see that $\lambda - \alpha_n$ and $\lambda - \alpha_{n+1}$ are two weights of $V_{D_{n+1}}$ restricting to $B_n$ as $\mu - \beta_n$, and by Freudenthal’s formula we see that $\mu - \beta_n$ has multiplicity one in $V_{B_n}(\mu)$. So $\mu - \beta_n$ appears also in a second irreducible summand, namely $V_{B_n}(\nu) = V_{B_n}(d\mu_1 + \mu_{n-1} + (b - 1)\mu_n)$ by lemma 2.1.3.

If $d = 0$, we have by lemma 1.3.12 that $V_{D_{n+1}}(\lambda)|_{B_n} = V_{B_n}(\mu) \oplus V_{B_n}(\nu)$.

If $d$ is not equal to zero, we see that $\lambda - \sum_{i=1}^n \alpha_i$ and $\lambda - \sum_{i=1}^{n-1} \alpha_i - \alpha_{n+1}$ are two weights restricting as $\gamma = \mu - \sum_{i=1}^n \beta_i$. Computing the multiplicities, and since we assumed $b \neq 0$, we have

<table>
<thead>
<tr>
<th>Case b = 0</th>
<th>Case b ≥ 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$mV_{B_n}(\mu)(\gamma)$</td>
<td>$i$</td>
</tr>
<tr>
<td>$mV_{B_n}(\nu)(\gamma)$</td>
<td>$i - 1$</td>
</tr>
<tr>
<td>$mV_{D_{n+1}}(\lambda)(\lambda - \sum_{j=1}^n \alpha_j)$</td>
<td>$i$</td>
</tr>
<tr>
<td>$mV_{D_{n+1}}(\lambda)(\lambda - \sum_{j=1}^{n-1} \alpha_j - \alpha_{n+1})$</td>
<td>$i$</td>
</tr>
<tr>
<td>Total multiplicity in $V_{D_{n+1}}(\lambda)$</td>
<td>$2i$</td>
</tr>
<tr>
<td>Difference between the multiplicities</td>
<td>1</td>
</tr>
</tbody>
</table>

which implies that for $d \neq 0$, there is a third irreducible summand.

In this last case, we deduce that:
1) When \( \lambda = b\lambda_n + \lambda_{n+1} \), the restriction of \( V_{D_{n+1}}(\lambda) \) to \( B_n \) consists exactly of the two irreducible summands \( V_{B_n}((b+1)\mu_n) \) and \( V_{B_n}(\mu_{n-1} + (b-1)\mu_n) \).

2) When \( \lambda = d\lambda_1 + b\lambda_n + \lambda_{n+1} \) for \( d > 0 \), there are at least three irreducible summands in the restriction of \( V_{D_{n+1}}(\lambda) \) to \( B_n \).
Index of notations

$K$ Algebraically closed field of characteristic zero
$\mathcal{L}$ Lie algebra
$\text{rad}(\mathcal{L})$ Radical of a Lie algebra
$\text{ad}_x$ Adjoint representation applied to $x \in \mathcal{L}$
$\mathfrak{h}$ or $\mathfrak{h}_L$ Cartan subalgebra of $\mathcal{L}$
$L_\alpha$ Root space for $\alpha$
$\alpha$ Root of $\mathcal{L}$ relative to $\mathfrak{h}$
$\mathfrak{h}^*$ Dual space of a Cartan subalgebra $\mathfrak{h}$
$\Phi$ or $\Phi_L$ Root system of the Lie algebra $\mathcal{L}$
$\langle - , - \rangle$ Scalar product on the euclidean space associated to the root system
$\langle - , - \rangle$ Product on the root system defined by $\langle \alpha , \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$
$\sigma_i$ Reflection through $\alpha_i$
$\kappa(-,-)$ Killing form
$\alpha_i$ Simple root
$t_\alpha$ Element of $\mathcal{L}$ such that $\alpha(h) = \kappa(t_\alpha, h)$ for every $h \in \mathfrak{h}$
$h_\alpha$ Element of $\mathcal{L}$ such that $h_\alpha = \frac{2\alpha}{\kappa(t_\alpha, t_\alpha)}$
$E_\mathbb{R}$ $\mathbb{R}$-space spanned by the elements of a root system $\Phi$
$\Delta$ Set of simple roots of a Lie algebra
$\prec$ Partial ordering on the euclidean space $E$ associated to the root system
$\Phi^+$ Set of positive roots of $\Phi$
$W$ Weyl group
$|\alpha|$ Length of $\alpha$
$N_L(H)$ Normalizer of a subalgebra $H$ in $\mathcal{L}$
$\mathfrak{b}$ or $\mathfrak{b}_L$ Borel subalgebra of $\mathcal{L}$
$\alpha_i$ Simple root
\[ \Omega(L) \] Universal enveloping algebra of \( L \)

\[ T(L) \] Tensor algebra of \( L \)

\[ e_\alpha \] Element of \( L_\alpha \) of the set of generators of \( L \)

\[ f_\alpha \] Element of \( L_{-\alpha} \) of the set of generators of \( L \)

\( \lambda \) Weight

\( \Lambda \) Set of weights of \( L \)

\( \lambda_i \) Fundamental dominant weight

\( \delta \) \[ \delta = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_i \lambda_i \]

\( V_\lambda \) Weight space for \( \lambda \)

\( \Lambda(V) \) Set of weights of an \( L \)-module \( V \)

\( v^+ \) Maximal vector

\( V_L(\lambda) \) Irreducible \( L \)-module with highest weight \( \lambda \)

\( m_V(\lambda) \) Multiplicity of the weight \( \lambda \) in the \( L \)-module \( V \)

\( \lambda|_X \) Restriction of the weight \( \lambda \) of \( L \) to a subalgebra \( X \)

\( V_L(\lambda)|_X \) Restriction of the \( L \)-module \( V_L(\lambda) \) to a subalgebra \( X \)

\( J \) Subset of a base \( \Delta \) of \( \Phi \)

\[ \Phi_J \] \[ \sum_{\alpha \in J} \mathbb{Z}\alpha \cap \Phi \text{ for } J \subset \Delta \]

\( a \) Subalgebra \( \langle e_\alpha, f_\alpha \mid \alpha \in J \rangle + \mathfrak{h} \) of \( L \) for \( J \subset \Delta \)

\( I_J \) Subalgebra \( \langle e_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_J \rangle \) for \( J \subset \Delta \)

\( P_J \) Parabolic subalgebra \( a \oplus I_J \)

\( a' \) Derived algebra of \( a \)

\( V_0(I_J) \) Set of elements of an \( L \)-module \( V \) on which the action of \( I_J \) is trivial

\( p \) Parabolic subalgebra

\( n \) Nilpotent radical of a parabolic subalgebra

\( \mathfrak{l} \) Levi subalgebra of a parabolic subalgebra
Bibliography


Appendix A

Table of types of simple Lie algebras

**Type $A_n$**

![Dynkin diagram of type $A_n$](image)

Figure A.1: Dynkin diagram of type $A_n$.

$$
\begin{pmatrix}
2 & -1 & 0 & \cdot & 0 & 0 \\
-1 & 2 & -1 & \cdot & 0 & 0 \\
0 & -1 & 2 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 2 & -1 \\
0 & 0 & 0 & \cdot & -1 & 2
\end{pmatrix}
$$

Cartan matrix of type $A_n$.

**Type $B_n$**

![Dynkin diagram of type $B_n$](image)

Figure A.2: Dynkin diagram of type $B_n$.

$$
\begin{pmatrix}
2 & -1 & 0 & \cdot & 0 & 0 \\
-1 & 2 & -1 & \cdot & 0 & 0 \\
0 & -1 & 2 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 2 & -2 \\
0 & 0 & 0 & \cdot & -1 & 2
\end{pmatrix}
$$

Cartan matrix of type $B_n$. 
Type $D_n$

Figure A.3: Dynkin diagram of type $D_n$.

\[
\begin{pmatrix}
2 & -1 & 0 & . & 0 & 0 & 0 \\
-1 & 2 & -1 & . & 0 & 0 & 0 \\
0 & -1 & 2 & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . \\
0 & 0 & 0 & . & 2 & -1 & -1 \\
0 & 0 & 0 & . & -1 & 2 & 0 \\
0 & 0 & 0 & . & -1 & 0 & 2
\end{pmatrix}
\]

Cartan matrix of type $D_n$.

Type $G_2$

Figure A.4: Dynkin diagram of type $G_2$.

\[
\begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}
\]

Cartan matrix of type $G_2$. 