MASTER PROJECT

Representations of $SL_2(K)$

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Introduction

The special linear group $\text{SL}_n(K)$ of degree $n$ over a field $K$ is defined as the kernel of the determinant morphism $\text{Det} : \text{GL}_n \to K^*$, i.e.

$$\text{SL}_n(K) = \{ A \in \text{GL}_n(K) : \text{Det}(A) = 1 \}.$$ 

The aim of this project is to study the representation theory of $\text{SL}_2(K)$ over an algebraically closed extension $L$ of $K$, where $K$ could either denote a finite field or $L$ itself, and then to study the decomposition of the tensor product of some particular $L\text{SL}_2(L)$-modules (the symmetric powers) in terms of indecomposables.

In the first chapter, we recall some basic properties of the special linear group and of modules in general. We also define the symmetric and alternating powers of the natural $L\text{GL}_n(K)$-module. Most of the proofs shall be omitted, but the reader can consult [Alp86], [ARO97] or [FH91] if interested.

In the second chapter, we consider $q = p^n$, where $p$ is a prime and $n \in \mathbb{N}^*$ a positive integer, and we give a complete set of non-isomorphic irreducible representations of $\text{SL}_2(q)$ over the algebraic closure of $F_p$. In order to do that, we start by finding a way to determine how many of them there are, following the theory of highest weights and highest weight vectors presented in [Ste68] rather than studying conjugacy classes, which could have been another approach to the problem. Finally, we construct this exact number of non-isomorphic irreducible representations.

In the third chapter, we let $K$ be an algebraically closed field of characteristic zero, momentarily forget the special linear group and focus our attention on the representation theory of the Lie algebra $\mathfrak{sl}_2(K)$. We first give a complete set of non-isomorphic irreducible finite-dimensional $\mathfrak{sl}_2(K)$-modules, recall Weyl’s theorem of complete reducibility and finally, prove the Clebsh-Gordan formula, which tells us how to decompose the tensor product of any two irreducible $\mathfrak{sl}_2(K)$-modules in terms of irreducibles. This chapter is mainly based on [Hum78] and [Maz10].

In the fourth chapter, we let $K$ be an algebraically closed field of any characteristic and study the representations of $\text{SL}_2(K)$ over $K$. In fact, we shall view $\text{SL}_2(K)$ with its structure of connected algebraic group and consider its so-called rational representations. In the case where $K$ has characteristic zero, we shall
see that the representation theory of $\text{SL}_2(K)$ is similar to the representation theory of the Lie algebra $\text{sl}_2(K)$. More precisely, the irreducibles are the same, every $\text{SL}_2(K)$-module is completely reducible and the Clebsh-Gordan formula still holds. On the other hand, in the case where $K$ has positive characteristic, a complete set of non-isomorphic irreducible rational $K\text{SL}_2(K)$-modules can be given using the set obtained in the second chapter. The references for this chapter are [Bor91], [Hum75] and [Spr81].

In the last chapter, we let $K$ be an algebraically closed field of positive characteristic and study the decomposition of the tensor product of any two symmetric powers in terms of indecomposable rational $K\text{SL}_2(K)$-modules. Our approach to the problem consists in finding such a decomposition as indecomposable polynomial $K\text{GL}_2(K)$-modules first and then in trying to deduce the desired result. Our reason for considering these polynomial $K\text{GL}_2(K)$-modules lies in the fact that a very important tool is available in this situation, called the Schur algebra. This chapter takes up the ideas of [Gre81] and [Mar08], principally concerning the study of the Schur algebra and the classification of the irreducible polynomial $K\text{GL}_n(K)$-modules in terms of partitions. We shall also refer to [Don98].
The aim of this chapter is to remind the reader of some basic knowledge, necessary to fully understand the project. As said in the introduction, we shall fix some notation and recall a few properties of the special linear group and of modules in general.

1.1 The special linear group

Let $K$ be a field and let $n \in \mathbb{N}^*$ be a positive integer. We first denote by $U(n)_K$ and $U(n)_K^-$ the upper and lower unipotent subgroups of $\text{SL}_n(K)$ respectively. In the same way, we let $T(n)_K$ be the subgroup of diagonal matrices. Finally, for every $a \in K$, we shall consider the elements $u(a) \in U(2)_K$, $u^-(a) \in U(2)_K^-$ and $h(a) \in T(2)_K^1$, respectively given as

\[ u(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad u^-(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad h(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \]

Lemma 1.1.1

The special linear group $\text{SL}_2(K)$ is generated by its two unipotent subgroups $U(2)_K$ and $U(2)_K^-$. In other words, we have

\[ \text{SL}_2(K) = \langle U(2)_K, U(2)_K^- \rangle. \]

Proof. Let $g = g_{ij}$ be an element of $\text{SL}_2(K)$. If $g_{12} \neq 0$, then $g$ can be written as $g = u^-((g_{22} - 1)g_{12}^{-1})u(g_{12})u^-(g_{11} - 1)g_{12}^{-1}$. If $g_{12} = 0$ but $g_{21} \neq 0$, then $g$ can be written as $g = u((g_{11} - 1)g_{21}^{-1})u^-(g_{21}^{-1})u(g_{11} - 1)g_{21}^{-1})$. Finally, if $g_{12} = g_{21} = 0$, then $g$ can be written as $g = u(-g_{11})u^-(g_{11}^{-1} - 1)u(1)u^-(g_{11} - 1)$. Hence any $g \in \text{SL}_2(K)$ can be written as a product of elements each belonging to $U(2)_K^-$ or $U(2)_K$ and thus the proof is complete.

\[ \square \]

1 The element $h(a)$ is only defined for $a \in K^*$. 
Remark 1.1.2
Return now to the case where \( n \in \mathbb{N}^* \) is an arbitrary positive integer and for \( 1 \leq i, j \leq n \), let \( E_{ij} \in M_{n \times n} \) be the matrix defined by
\[
(E_{ij})_{\mu\nu} = \begin{cases} 
0 : (i, j) = (\mu, \nu), \\
1 : \text{otherwise}
\end{cases}
\]

A transvection of \( \text{SL}_n(K) \) is a matrix of the form \( I_n + \lambda E_{ij} \), where \( \lambda \in K \) and \( 1 \leq i \neq j \leq n \). It is clear that such matrices are in \( \text{SL}_n(K) \) and in fact, Lemma 1.1.1 is a special case of a more general result, which states that the special linear group \( \text{SL}_n(K) \) is generated by its transvections.

Now consider the case where \( K \) is a finite field, that is, \( K = \mathbb{F}_q \) for some \( q = p^m \), with \( p \) a prime and \( m \in \mathbb{N}^* \) a positive integer. Then the reader can easily check that the order of the general linear group of rank \( n \) over \( K \) is given as
\[
|\text{GL}_n(q)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - 1).
\]
As a direct consequence, since the special linear group \( \text{SL}_n(q) \) is defined as the kernel of the determinant morphism \( \text{Det} : \text{GL}_n(q) \to \mathbb{F}_q^* \), we obviously have
\[
|\text{SL}_n(q)| = q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - 1) \quad (1.1)
\]
and an easy computation yields the following result.

Lemma 1.1.3
Both \( U(n)_{\mathbb{F}_q} \) and \( U^-(n)_{\mathbb{F}_q} \) have order \( q^{n(n-1)/2} \). In particular, they are \( p \)-Sylow subgroups of \( \text{SL}_n(q) \).

1.2 Representations of algebras

Let \( K \) be an algebraically closed field and let \( A \) be a finite-dimensional \( K \)-algebra. Throughout this report, we shall only consider finite-dimensional left \( A \)-modules, unless otherwise indicated. Nevertheless, most of the following results still hold in a more general setting. We refer to [Alp86] and [ARO97] for more details.

Lemma 1.2.1 (Schur’s Lemma)
Let \( V, W \) be irreducible \( A \)-modules and let \( \phi : V \to W \) be a morphism of \( A \)-modules. Then \( \phi \) is either trivial or an isomorphism. Furthermore,
\[
\text{Hom}_A(V, V) \cong K.
\]

Now recall that a composition series for an \( A \)-module \( V \) is a finite sequence of submodules \( V = V_0 \supset V_1 \supset \cdots \supset V_r = 0 \) such that \( V_i/V_{i+1} \) is irreducible, for every \( 0 \leq i \leq r - 1 \).
**Theorem 1.2.2** (Jordan-Hölder Theorem)
Assume that \( V = W_0 \supset W_1 \supset \ldots \supset W_{s-1} \supset W_s = 0 \) is another composition series for \( V \). Then \( r = s \) and for any irreducible \( A \)-module \( S \), we have

\[
\{i : V_i / V_{i+1} \cong S\} = \{i : W_i / W_{i+1} \cong S\}.
\]

*Proof.* See [ARO97, Theorem 1.2, pp. 3]. \( \square \)

Let \( V \) and \( S \) be two \( A \)-modules, with \( S \) irreducible, and consider any composition series \( V = V_0 \supset V_1 \supset \ldots \supset V_{r-1} \supset V_r = 0 \) for \( V \). Then the *composition multiplicity of \( S \) in \( V \) * (well-defined thanks to Theorem 1.2.2) is the non-negative integer

\[
[V : S] = \#\{i : V_i / V_{i+1} \cong S\}.
\]

**Theorem 1.2.3** (Krull-Schmidt Theorem)
Any \( A \)-module is isomorphic to a direct sum of indecomposable \( A \)-modules. Moreover, the latter are uniquely determined, up to order and isomorphism.

Finally, recall that an \( A \)-module is said to be *completely reducible* (or *semisimple*) if it is isomorphic to a direct sum of irreducible \( A \)-modules. Again, the decomposition has to be unique (up to order and isomorphism) by Theorem 1.2.2 or Theorem 1.2.3. The following result gives a characterization of such a module. Its proof can be found in [Alp86, Proposition 2, pp. 2-3].

**Lemma 1.2.4**
An \( A \)-module is completely reducible if and only if every \( A \)-submodule is a direct summand.

Now the *radical* of \( A \) (denoted by \( \text{Rad}(A) \)) is the ideal consisting of all the elements of \( A \) which annihilate every irreducible \( A \)-module. The reader can consult [Alp86, Proposition 4, pp. 3-5] for a proof of the following result.

**Lemma 1.2.5**
The radical of \( A \) is equal to each of the following:

(a) The smallest submodule of \( A \) whose corresponding quotient is semisimple;

(b) The intersection of all maximal submodules of \( A \);

(c) The largest nilpotent ideal of \( A \).

We say that \( A \) is *semisimple* if \( \text{Rad}(A) = 0 \). By Lemma 1.2.5 (a), the \( K \)-algebra \( A \) is semisimple if and only if it is semisimple as an \( A \)-module. Moreover, we have that \( A \) is semisimple if and only if every \( A \)-module is semisimple since we only consider finite-dimensional \( A \)-modules.
Now let $G$ be a finite group and let $KG$ denote the group algebra of $G$. Then the reader can find a proof of the following result (in characteristic zero) in [FH91, Corollary 2.18, pp. 17] or [JL93, Theorem 11.9, pp. 100].

**Lemma 1.2.6**

*If the group algebra $KG$ of $G$ is semisimple, then every irreducible $KG$-module $S$ appears exactly $\dim(S)$ times in any direct sum of irreducible $KG$-modules.*

The trivial representation of $G$ over $K$ can be extended to a morphism of $K$-algebras $1 : KG \to K$ as follows: for every $\kappa = \sum_{g \in G} \alpha_g g$ in $KG$, set $1(\kappa) = \sum_{g \in G} \alpha_g g$. Its kernel is an ideal of $KG$, called the augmentation ideal of $KG$ and denoted by $\Delta(G)$. The following is a basic but very important result in the representation theory of finite groups. We shall give some ideas of the proof and refer to [Alp86, Theorem 1, pp. 12-13] for the details.

**Theorem 1.2.7** (Maschke’s Theorem)

*Let $G$ be a finite group. Then the group algebra $KG$ of $G$ is semisimple if and only if the characteristic of $K$ does not divide the order of $G.*

**Proof.** The augmentation ideal $\Delta(G)$ of $KG$ is a submodule of $KG$ verifying $KG/\Delta(G) \cong K$. Now if \text{char}(K) divides $|G|$, the sum $\sum G$ of all elements in $G$ belongs to $\Delta(G)$ and the reader can easily check that $KGG$ is a $KG$-submodule of $KG$, also isomorphic to the trivial $KG$-module. Hence when the series

$$KG \supset \Delta(G) \supset KG$$

is refined to a composition series, the trivial $KG$-module appears twice and thus $KG$ is not semisimple by Lemma 1.2.6. Conversely, if \text{char}(K) does not divide $|G|$, then $|G|$ is an invertible element of $K$. Hence if $V$ is any $KG$-submodule and $W$ is a $KG$-submodule of $V$, then one can show that $W$ is a direct summand of $V$ by considering a projection $\pi_W$ onto $W$ and then defining a new $KG$-morphism $\pi' : V \to V$ by

$$\pi'(v) = \frac{1}{|G|} \sum_{g \in G} g\pi_W(g^{-1}v), \quad v \in V.$$  

\[ \Box \]

We now establish a property of irreducible representations of abelian groups over an algebraically closed field $K$ of any characteristic.

**Lemma 1.2.8**

*Let $G$ be an abelian group and let $V$ be an irreducible $KG$-module. Then $V$ is one-dimensional.*

**Proof.** Let $g \in G$. Since $G$ is abelian, the endomorphism $v \mapsto gv$ is a morphism of $KG$-modules and thus by Lemma 1.2.1, there exists $\lambda(g) \in K^*$ such that $gv = \lambda(g)v$ for every $v \in V$. Therefore, every $K$-subspace of $V$ is stable under the action of $G$ and thus the result follows.

\[ \Box \]
In the next chapter, we shall work with the unipotent subgroups of \( SL_2(F_q) \), where \( m \in \mathbb{N}^* \) and \( q = p^m \), which are \( p \)-groups by Lemma 1.1.3. Therefore, it will be interesting to give a few properties of representations of \( p \)-groups over algebraically closed fields of characteristic \( p \). Let then \( K \) denote such a field and let \( G \) be a finite \( p \)-group.

**Lemma 1.2.9**

The following assertions hold:

(a) Every \( KG \)-module \( 0 \neq V \) contains non-zero vectors fixed by \( G \),

(b) Every irreducible \( KG \)-module is trivial,

(c) The ideals \( \Delta(G) \) and \( \text{Rad}(KG) \) of \( KG \) are equal.

**Proof.**

(a) We proceed using induction on \( |G| \). In the case where \( |G| = 1 \), the result is obvious. Also if \( G = C_p \) is cyclic of order \( p \), then any irreducible \( KG \)-submodule \( W \) of \( V \) is one-dimensional by Lemma 1.2.8 and even trivial. Indeed, let \( \lambda : G \to K^* \) be the representation afforded by \( W \). Then \( \lambda^p = \lambda = 1 \) and thus the result for \( |G| = p \). Assume then that \( |G| = p^n \) for some positive integer \( n > 1 \). Since \( G \) is a \( p \)-group, it contains a normal subgroup \( N \) of index \( p \) and by the induction hypothesis, the subspace \( V^N \subseteq V \) of \( N \)-fixed vectors in \( V \) is non-zero. Also \( G/N \) acts on \( V^N \) and using the induction hypothesis one more time, we can affirm that there exists a non-zero vector \( v \in V^N \) fixed by \( G/N \).

(b) Let \( V \) be an irreducible \( KG \)-module. By (a), \( V \) contains a non-zero vector \( v \) fixed by \( G \) and thus contains the trivial \( KG \)-module. Now since \( V \) was assumed irreducible, we get the desired result.

(c) The ideal \( \Delta(G) \) is clearly maximal since its codimension in \( KG \) is one and hence \( \text{Rad}(KG) \subseteq \Delta(G) \) by condition (b) of Lemma 1.2.5. Now consider any composition series for \( KG \). Then \( G \) acts trivially on every composition factor by (b) and thus \( \Delta(G) \) is nilpotent. Therefore, \( \Delta(G) \) is contained in \( \text{Rad}(KG) \) by condition (c) of Lemma 1.2.5 and the result follows.

Finally, we recall a very important result in the representation theory of finite groups, which provides a way of finding a complete set of non-isomorphic irreducible \( KSL_2(q) \)-modules, where \( K \) denotes the algebraic closure of \( F_p \). Its proof can be found in [Alp86, pp. 14-20].

**Theorem 1.2.10**

The number of irreducible \( KG \)-modules equals the number of conjugacy classes of \( G \) of elements of order not divisible by the characteristic of \( K \).
1.3 Duality

Let $A$ be a finite-dimensional $K$-algebra and let $V$ be a left $A$-module. Then its linear dual $V^* = \text{Hom}_K(V, K)$ can be provided with the structure of a right $A$-module. Indeed, if $a \in A$ and $\theta \in V^*$, then define the element $\theta \cdot a$ of $V^*$ such that

$$(\theta \cdot a)(v) = \theta(av), \ v \in V. \quad (1.2)$$

However, it will be interesting to provide $V^*$ with the structure of a left $A$-module. Suppose that $\sigma : A \to A$ is an involutive anti-automorphism of $A$, i.e. a $K$-linear map such that $\sigma(1) = 1$, $\sigma^2(x) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$, this for every $x, y \in A$. Then $V^*$ is a left $A$-module, with corresponding action

$$(a \cdot \theta)(v) = \theta(\sigma(a)v), \ a \in A, \ \theta \in V^* \text{ and } v \in V,$$

called the contravariant dual of $V$ modulo $\sigma$ and often denoted by $V^{\sigma}$.

Example:

Let $G$ be a group and let $V$ be a left $KG$-module. Then we usually consider the anti-automorphism $^{-1} : KG \to KG$ sending any element $g \in G$ to its inverse $g^{-1}$ (extended linearly to $KG$) in order to give $V^*$ the structure of a left $KG$-module.

Now the following results hold for both kinds of duals presented before, but we only show them for contravariant ones and so we drop the left or right assumptions. First let $\sigma : A \to A$ be an anti-automorphism of $A$. Then if $f : V \to W$ is a morphism of $A$-modules, its transpose $f^\sigma : W^\sigma \to V^{\sigma}$, defined by $f^\sigma(\theta)(v) = \theta(f(v))$ for any $v \in V$, is a morphism of $A$-modules as well. More generally, this construction leads to the linear isomorphism

$$\text{Hom}_A(V, W) \cong \text{Hom}_A(W^{\sigma}, V^{\sigma}). \quad (1.3)$$

In fact, all the usual properties of duality carry over to $A$-modules. For example, if $V$ is an $A$-module, then the natural isomorphism $V^{\sigma} \cong V$ of $K$-spaces is a morphism of $A$-modules. Similarly, if $V$ and $W$ are two $A$-modules, the isomorphisms $(V \oplus W)^\sigma \cong V^{\sigma} \oplus W^{\sigma}$ and $(V \otimes W)^\sigma \cong V^{\sigma} \otimes W^{\sigma}$ are isomorphisms of $A$-modules. Finally, consider a short exact sequence of $A$-modules $0 \to W \to V \to V/W \to 0$.

Lemma 1.3.1

The induced sequence $0 \to (V/W)^\sigma \to V^\sigma \to W^\sigma \to 0$ is a short exact sequence of $A$-modules. In particular, $V$ is irreducible if and only if $V^\sigma$ is irreducible.

Proof. First notice that the $A$-submodule $Z = \{\theta \in V^\sigma : \theta|_W = 0\}$ of $V^\sigma$ can be identified with $(V/W)^\sigma$ using the map $\phi : Z \to (V/W)^\sigma$, defined such that $\phi(\theta)(v + W) = \theta(v)$ for every $\theta \in V^\sigma$ and every $v \in V$. Now the $A$-morphism $\psi : V^\sigma \to W^\sigma$ sending $\theta \in V^\sigma$ to $\theta|_W$ is surjective and has kernel $Z$, whence the result. \[ \square \]
1.4 Symmetric and exterior powers

First recall that if \( G \) is an infinite group, then the group algebra \( KG \) of \( G \) consists of all finite linear combinations of elements each belonging to \( G \), with addition and multiplication as in the finite case. Let then \( n \in \mathbb{N}^+ \) be a positive integer and let \( L \) be a field extension of \( K \). Denote by \( E = E_L(n) \) the \( n \)-dimensional vector space of column vectors over \( L \), i.e. \( E = \langle e_1, \ldots, e_n \rangle_L \), where \( (e_i)_j = \delta_{ij} \) for \( 1 \leq i, j \leq n \). Now the general linear group \( \text{GL}_n(K) \) acts naturally on \( E \), which becomes an \( \text{GL}_n(K) \)-module with action as follows: for \( 0 \leq k \leq n \) and \( g = g_{ij} \in \text{GL}_n(K) \), we have

\[
g \cdot e_k = \sum_{r=1}^{n} g_{rk} e_r.
\]

Furthermore, recall that if \( G \) denotes a group and \( V \) and \( W \) are two \( LG \)-modules, then \( V \otimes W \) is an \( LG \)-module with corresponding action given on generators as

\[
g \cdot (v \otimes w) = (gv) \otimes (gw), \quad (1.4)
\]

Consequently, the tensor algebra \( T(E) = \bigoplus_{r \in \mathbb{N}} E^\otimes r \) of \( E \) admits the structure of an \( \text{GL}_n(K) \)-module and it is easy to see that the ideals \( I \) generated by all differences of products \( x \otimes y - y \otimes x \) \((x, y \in E)\) and \( J \) generated by all elements of the form \( x \otimes x \) \((x \in E)\) are \( \text{GL}_n(K) \)-submodules of \( T(E) \). This immediately leads to the definitions of the symmetric and exterior algebras on \( E \), respectively given as

\[
S(E) = T(E)/I \quad \text{and} \quad \Lambda(E) = T(E)/J.
\]

Now it is possible to view the symmetric algebra \( S(E) \) on \( E \) as the polynomial algebra \( S(E) = L[e_1, \ldots, e_n] \), on which \( \text{GL}_n(K) \) acts as algebra automorphisms, extending the action on \( E \) according to the rule

\[
g \left( \prod_{k=1}^{n} e_{ik}^{i_k} \right) = \prod_{k=1}^{n} g(e_k)^{i_k}, \quad (1.5)
\]

for any \( g \in G \) and any \( i_1, \ldots, i_n \in \mathbb{N} \). With this vision, for \( r \in \mathbb{N} \), we define the \( r \)-th symmetric power \( S^r E \) of \( E \) to be the \( r \)-th homogeneous subspace of \( S(E) \), i.e. the \( L \)-subalgebra of \( S(E) \) spanned by elements of the form \( e_{i_1} \cdots e_{i_r} \), where \( 1 \leq i_1 \leq \ldots \leq i_r \leq n \). We then have an obvious grading of \( S(E) \) as an \( L \)-algebra

\[
S(E) \cong \bigoplus_{r \in \mathbb{N}} S^r E. \quad (1.6)
\]

Lemma 1.4.1

For every \( r \in \mathbb{N} \), the \( L \)-space \( S^r E \) is an \( \text{GL}_n(K) \)-submodule of \( S(E) \). Thus the grading \( (1.6) \) is a decomposition as \( \text{GL}_n(K) \)-modules.

Proof. Consider \( 0 \leq i_1, \ldots, i_r \leq n \) and \( g \in \text{GL}_n(K) \). Then since \( ge_{i_k} \in E \) for any \( 1 \leq k \leq n \) and since the action of \( G \) on \( S(E) \) is given by Equation (1.5), we clearly have \( g(e_{i_1} \cdots e_{i_r}) \in S^r E \) and the \( L \)-algebra \( S^r E \) is stable under the action of \( G \).

\( \square \)
In a similar way, for a given $r \in \mathbb{N}$, it is possible to define the $r$th exterior power $\Lambda^r E$ of $E$ as the $L$-subalgebra of $\Lambda(E)$ spanned by the elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_r}$, where $1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq n$ and $x \wedge y$ denotes the image of $x \otimes y$ under the quotient map $\pi : T(E) \to \Lambda(E)$. Again, we have the following grading of $\Lambda(E)$ as an $L$-algebra

$$\Lambda(E) \cong \bigoplus_{r \in \mathbb{N}} \Lambda^r E$$

and the following assertion holds.

**Lemma 1.4.2**

For every $r \in \mathbb{N}$, the $L$-space $\Lambda^r E$ is an $LGL_n(K)$-submodule of $\Lambda(E)$. Thus the grading (1.7) is a decomposition as $LGL_n(K)$-modules.
Throughout this chapter, we let $p$ be a prime and $q = p^n$, where $n$ is a positive integer. We then let $K$ be an infinite field containing $\mathbb{F}_p$ and denote by $G$ the special linear group of degree 2 over $\mathbb{F}_q$, which has order $|G| = (q - 1)q(q + 1)$ by Equation (1.1). We also let $U$, $U^-$ and $H$ respectively denote the three subgroups $U(2)_{\mathbb{F}_q}$, $U(2)_{\mathbb{F}_q}^-$ and $T(n)_{\mathbb{F}_q}$ introduced in the previous chapter and define the elements

$$U = \sum_{a \in \mathbb{F}_q} u(a) \in KU, \quad U^- = \sum_{a \in \mathbb{F}_q} u^-(a) \in KU^-.$$ 

Finally, we denote by $w$ the element of $G$ given as

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

The aim of this chapter consists in giving a complete set of non-isomorphic irreducible $KG$-modules. In order to do this, we choose to view $G$ with its Chevalley group structure and come up with a classification in terms of highest weights. Another approach would have been to use Theorem 1.2.10 together with the well-known table of conjugacy classes of $G$, which can be found in [FH91, ï¿125.2, pp.71], for example.

2.1 The theory of highest weights

We now introduce the definition of highest weight and highest weight vector and study their relationship with the irreducible $KG$-modules. All the arguments can be translated into the language of Chevalley groups and so are still valid in a more general context. Indeed, this section is basically a simplified version of [Ste68, pp. 230-236], in which the case of any Chevalley group of rank one is treated.
**Definition 2.1.1**

A non-zero vector \( v \) in a KG-module \( V \) is called a highest weight vector if it satisfies the following properties:

(a) \( xv = v \) for every \( x \in U \),

(b) \( hv = \lambda(h)v \) for every \( h \in H \) and \( \lambda \) some character on \( H \),

(c) \( Uwv = \mu v \) for some \( \mu \in K \).

**Remark 2.1.1**

In the situation above, the couple \( (\lambda,\mu) \) is called the weight of \( v \). At this point, the reason why condition (c) was introduced is quite unclear. However, we shall see later that it allows us to distinguish the ‘smallest’ KG-module from the ‘largest’ and replaces a density argument used in the case where \( K \) is infinite.

We might now ask about existence and uniqueness of highest weight vectors in an arbitrary KG-module, but first observe that for every \( a \in F_q^\ast \), we have

\[
wu(a)wv = u(-a^{-1})hv = \lambda(-a^{-1})v.
\]

Thus, by Lemma 1.1.3, \( U \) is a p-group and hence by Lemma 1.2.9 (a), there exists a non-zero vector \( v \in V \) fixed by \( U \). Now since \( H \) normalizes \( U \), for any \( h \in H \) and \( x \in U \), one has \( xhv = hh^{-1}xhv = hv \), i.e. \( V^U \) is a KH-module. But \( H \) is abelian and so by Lemma 1.2.8, we can find \( v \in V^U \) satisfying conditions (a) and (b) of Definition 2.1.1. Now if \( Uwv = 0 \), then \( v \) is a highest weight vector of weight \( (\lambda,0) \). If on the contrary \( Uwv \neq 0 \), set \( v_1 = Uwv \). Then for any \( x \in U \), \( xv_1 = xUwv = Uwv = v_1 \), and thus \( v_1 \) satisfies condition (a) of Definition 2.1.1. Morever, for any \( h \in H \), \( hv_1 = hUwv = Uwh^{-1}v = \lambda(h^{-1})v_1 \), and thus condition (b) of Definition 2.1.1 holds replacing \( \lambda \) by \( w\lambda \), where \( w\lambda \) is the character on \( H \) defined by \( w\lambda(h) = \lambda(h^{-1}) \), for every \( h \in H \). Finally, Equation (2.1) yields

\[
Uwv_1 = -Uv + \sum_{a \in F_q^\ast} h(a)Uwv(a)v = \left( \sum_{a \in F_q^\ast} w\lambda(h(a)) \right) v_1.
\]

Consequently, \( v_1 \) satisfies the third and last condition of Definition 2.1.1 and is therefore a highest weight vector of weight \( (w\lambda,\mu) \).

In order to prove the second statement, let \( v \) be a highest weight vector of weight \( (\lambda,\mu) \). It is easy to verify that \( G \) can be written as \( G = U^{-B} \cup wB \).
where \( B = UH \) denotes the subgroup of upper triangular matrices in \( G \) (this decomposition still holds for any Chevalley group of rank one). Therefore, we have

\[
KGv = KU^-Bv \cup KwBv = KU^-v \cup Kwv,
\]

where for any subset \( X \) of \( V \), we let \( KX \) denote the \( K \)-subspace of \( V \) generated by all finite linear combinations of elements each belonging to \( X \). It only remains to show that \( wv \in KU^-v \).

Now for every \( a \in \mathbb{F}_q^\ast \), we have

\[
u(a)wv = w^2u(a)w^{-1}v = wwu(a)w^{-1}v = wu(-a^{-1})h(-a^{-1})w^{-1}u(-a^{-1})v = wu(-a^{-1})h(-a^{-1})w^{-1}v \]

\[
= wu(-a^{-1})w^{-1}h(-a)v = \lambda(h(-a))wu(-a^{-1})w^{-1}v.
\]

and thus using the equality \( wu(a)w = u(-a^{-1})h(-a^{-1})wu(-a^{-1}) \) one more time yields

\[
\mu v = wv + \sum_{a \in \mathbb{F}_q^\ast} \lambda(h(a))wu(a^{-1})w^{-1}v. \tag{2.2}\]

\[\square\]

**Theorem 2.1.3 (Uniqueness)**

Let \( V \) be an irreducible \( KG \)-module and let \( v \in V \) be a highest weight vector. Then the line \( Kv \) is the unique line in \( V \) fixed by \( U \).

**Proof.** Set \( V' = Kv \) and \( V'' = \Delta(U^-)v \), which are \( K \)-subspaces of \( V \). As we know, \( KU^-/\Delta(U^-) \cong K \) as a \( KU^- \)-module and thus \( KU^- = \Delta(U^-) \oplus K \) as a \( K \)-vector space. Now \( KU^-v = V \) by Theorem 2.1.2, so

\[
V = V' + V''.
\]

Moreover, if \( x \in V' \cap V'' \), then \( x = \alpha v \) for some \( \alpha \in K \) and \( x = zv \) for some \( z \in \Delta(U^-) \). Then applying \( z \) recursively to \( x \) and using Lemma 1.2.9 (c), one gets \( x = 0 \). Consequently, \( V \) can be written (as a \( K \)-vector space) as

\[
V = V' \oplus V''.
\]

Assume now that there exists some \( U \)-fixed vector \( v_1 \in V \setminus V' \). Without loss of generality, one can suppose that \( v_1 \in V'' \) and since \( H \) is abelian, that \( v_1 \) is an eigenvector for \( H \). Moreover, we have \( \mu wv = wU^-v_1 = wU^-v_1yv \) for some \( y \in \Delta(U^-) \) and hence \( v_1 \) is a highest weight vector of weight \( (\lambda,0) \), where \( \lambda \) is a character on \( H \). By Theorem 2.1.2, one gets

\[
V = KU^-v_1 \subset KU^-V'' = V'',
\]

which is a contradiction. \[\square\]
By Theorem 2.1.3, any irreducible $KG$-module $V$ has a unique highest weight vector (modulo a scalar multiple) and hence $V$ determines uniquely its highest weight $(\lambda, \mu)$. We can then talk about the highest weight of an irreducible $KG$-module without ambiguity. Now it is obvious that two isomorphic $KG$-modules have same highest weight. In order to show that the converse is true, we use a density argument which also appears in the case where $K$ is infinite.

**Theorem 2.1.4**

Any two irreducible $KG$-modules having the same highest weight are isomorphic.

**Proof.** Consider two irreducible $KG$-modules $V_1$ and $V_2$ having the same highest weight $(\lambda, \mu)$ and let $v_1$ and $v_2$ be highest weight vectors of $V_1$ and $V_2$ respectively. If $v = v_1 + v_2 \in V_1 \oplus V_2$, Theorem 2.1.2 yields

$$KGv = KUv.$$  

Now $v_2 \not\in KUv$, since otherwise we could write $v_2 = \alpha v + v''$, with $\alpha \in K$ and $v'' \in \Delta(U^-v)$ and then projecting on $V_1$ and $V_2$ would yield $0 = \alpha = 1$. Therefore, the projection $\pi_1 : KGv \rightarrow V_1$ is injective. Furthermore, Theorem 2.1.2 yields $KGv_1 = V_1$ and consequently $\pi_1$ is surjective as well. The same argument works for the projection $\pi_2 : KGv \rightarrow V_2$ and so

$$V_1 \xrightarrow{\cong} KGv \xrightarrow{\cong} V_2.$$  

We now establish the main result of this section, which gives us an upper bound for the number of irreducible $KG$-modules. Moreover, it tells us information on the possible values of the highest weights of these modules.

**Theorem 2.1.5**

Let $(\lambda, \mu)$ be the highest weight of an irreducible $KG$-module. If $\lambda \neq 1$, then it completely determines $\mu$. If on the contrary $\lambda = 1$, then $\mu = 0$ or $-1$. Therefore, there are at most $|H| + 1 = q$ non-isomorphic irreducible $KG$-modules.

**Proof.** First consider a one-dimensional irreducible $KG$-module $V$. Then $V$ is irreducible as a $KU$-module (and as a $KU^-$-module as well) and by Lemma 1.2.9 (b), $V$ is trivial as a $KU$-module (and as a $KU^-$-module as well). Using Lemma 1.1.1, one concludes that $G$ acts trivially on $V$ and hence $V$ has highest weight $(\lambda, \mu) = (1, 0)$. Now if dim$(V) > 1$, write $V = V' \oplus V''$ as in the proof of Theorem 2.1.3 (clearly, $V'' \neq 0$ since dim$(V) > 1$). Then since $U^-$ fixes $V''$, it must fix a line in it and by Theorem 2.1.3 applied to $U^-$ instead of $U$, this line is uniquely determined in $V$. Now since $U^-$ fixes $wv$, we get $wv \in V''$ and projecting Equation (2.2) onto $V'$ yields

$$\mu = \sum_{h \in H} \lambda(wh^{-1}w).$$  

Hence if $\lambda = 1$ and dim$(V) = 1$, then $\mu = 0$. If $\lambda = 1$ and dim$(V) > 1$, then Equation (2.3) yields $\mu = -1$. Finally, if $\lambda \neq 1$, then dim$(V) > 1$ and thus by Equation (2.3), $\mu$ is determined by $\lambda$.  

\hfill $\Box$
2.2 Irreducible \( K \text{SL}_2(p) \)-modules

We first consider the special case where \( q = p \) is a prime and start by proving a result of linear algebra, which we shall use several times in the remainder of the project.

Lemma 2.2.1
Let \( K \) be a field, \( V \) a \( K \)-vector space and \( U \) a subspace of \( V \). In addition suppose that \( v_0, \ldots, v_m \in V \) are such that \( v_0 + tv_1 + \cdots + t^m v_m \in U \) for \( m + 1 \) values of \( t \in K \), then \( v_i \in U \) for \( 0 \leq i \leq m \).

Proof. Ab absurdo, assume that there exists some \( 0 \leq i \leq m \) such that \( v_i \notin U \), i.e. such that \( v_i + U \notin V/U \) is non-zero. Then one can choose \( \phi \in (V/U)^* \) such that \( \phi(v_i + U) \neq 0 \). Lifting \( \phi \) to \( V \) gives us a new linear functional \( \tilde{\phi} \in V^* \) verifying \( \tilde{\phi}(v) = \phi(v + U) \) for every \( v \in V \). Clearly, \( \tilde{\phi}(v_i) \neq 0 \). But then applying \( \tilde{\phi} \) to \( v_0 + tv_1 + \cdots + t^m v_m \in U \) yields

\[
\tilde{\phi}(v_0) + t\tilde{\phi}(v_1) + \cdots + t^m \tilde{\phi}(v_m) = 0,
\]
this for every \( t \in K \). But the polynomial in Equation (2.4) is of degree \( m \) in the indeterminate \( t \) and has \( m + 1 \) roots, whence \( \tilde{\phi}(v_i) = 0 \), a contradiction. □

Return to the case where \( K \) is the algebraic closure of \( \mathbb{F}_p \) and let \( E = E(2) \) denote the two-dimensional vector space over \( K \) having \( K \)-basis \( \{e_1, e_2\} \). For any \( 0 \leq r \leq p - 1 \), the \( r \)-th symmetric power \( S^r E \) on \( E \) is a \( K \text{GL}_2(q) \)-module by Lemma 1.4.1 and so is a \( K \text{SL}_2(q) \)-module as well. We write \( V(r) = S^r E \) and recall that a \( K \)-basis of \( V(r) \) is given as \( \{e_1^r, e_1^{r-1}e_2, \ldots, e_1e_2^{r-1}, e_2^r\} \).

Finally, let \( \mu, \nu \in \mathbb{N} \) be two non-negative integers with \( \nu \leq \mu \). Then we shall write \( C_{\mu}^\nu \) for the binomial coefficient

\[
C_{\mu}^\nu = \binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu - \nu)!}.
\]

Lemma 2.2.2
The vector \( e_1^r \) generates \( V(r) \) as a \( KU^- \)-module.

Proof. For any \( a \in \mathbb{F}_p \), we have

\[
u^{-}(a)e_1^{r} = (u^{-}(a)e_1)^{r} = (e_1 + ae_2)^{r} = \sum_{i=0}^{r} C_{i}^{\nu} a^{\nu-i} e_1^{r-i} e_2^{r-i}.
\]
Now since \( 0 \leq r \leq \mu - 1 \), the binomial coefficients \( C_{i}^{\nu} \) are non-zero and then Lemma 2.2.1 yields \( e_1^{r} e_2^{r-i} \in KU^- e_1, \) for every \( 0 \leq i \leq r \). □
Lemma 2.2.3
The line $K e_1^r$ is the only $U$-invariant line in $V(r)$.

Proof. Let $v \in V(r) \setminus K e_1^r$, i.e. $v = \sum_{i=0}^{r} \alpha_i e_1^r e_2^{r-i}$, where $\alpha_i \neq 0$ for at least one $i$ different from $r$. Then for any $a \in \mathbb{F}_p$, we have

$$u(a)v = \sum_{i=0}^{r} \alpha_i u(a)(e_1^r e_2^{r-i})$$

$$= \sum_{i=0}^{r} \alpha_i (ae_1 + e_2)^{r-i}$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{r-i} \alpha_i a^j C_{j}^{r-i} e_1^{i+j} e_2^{r-(i+j)}$$

$$= \sum_{i=0}^{r} a^i \left( \sum_{j=0}^{r-i} \alpha_j C_{j}^{r-j} e_1^{i+j} e_2^{r-(i+j)} \right)_{v_i}$$

and it only remains to check that at least two $v_i$'s are non-zero. Indeed, by Lemma 2.2.1, this would imply that $K U v$ is not one-dimensional. Let then $j_0$ be the smallest integer satisfying $\alpha_{j_0} \neq 0$. Since $C_{\nu}^{\mu}$ is non-zero for every $0 \leq \nu \leq \mu \leq r$, the vector $v_i$ is non-zero for every $0 \leq i \leq r - j_0$. We then conclude using the fact that $j_0 < r$.

We are now able to prove the main result of this section, which provides a complete description of the irreducible $KSL_2(p)$-modules.

Theorem 2.2.4
For every $0 \leq r \leq p - 1$, the $KSL_2(p)$-module $V(r)$ is irreducible. Moreover, the list $\{V(r)\}_{r=0, \ldots, p-1}$ forms a complete set of non-isomorphic $KSL_2(p)$-modules.

Proof. Let $0 \leq r \leq p - 1$ be fixed and consider a non-zero $KSL_2(p)$-submodule $V$ of $V(r)$. Clearly, $V$ is a $K U$-module and therefore contains an irreducible $K U$-submodule $W$. However, by Lemma 1.1.3, $U$ is a $p$-group and so it follows from Lemma 1.2.9 (b) that $W$ is trivial as a $K U$-module, i.e. $V$ contains a $U$-invariant line $K v$. This line is obviously a $U$-invariant line in $V(r)$ and hence by Lemma 2.2.3, one can set $v = e_1^r$. Finally, by Lemma 2.2.2, we have

$$V \supset K U^{-} e_1^r = V(r),$$

which concludes the proof of the first affirmation. Now by Theorem 2.1.5, there are at most $p$ non-isomorphic irreducible $KSL_2(p)$-modules. But since the $V(r)$'s have different dimensions, they clearly are distinct from each other and so the second statement directly follows from the first.
2.3 Irreducible $KSL_2(q)$-modules

We now return to the general case $G = SL_2(q)$ and give a classification of the irreducible $KG$-modules. First observe that since $SL_2(p) \subset G$, the $KG$-modules $K, E, S^2E, \ldots, S^{p-1}E$ are irreducible, since they were irreducible as $KSL_2(p)$-modules by Theorem 2.2.4. However, Theorem 2.1.5 tells us that we still have $q - p$ irreducible $KG$-modules to determine. Now if $r \geq p$, the module $S^rE$ is not irreducible in general. Indeed, consider $q$ modules by Theorem 2.2.4. However, Theorem 2.1.5 tells us that we still have $2K, E, S$ any $g \in G$, in order to find the remaining irreducible is not irreducible. Consequently, it seems that we need to look somewhere else in order to find the remaining irreducible $KG$-modules.

2.3.1 The Frobenius twist of a $KSL_2(q)$-module

The Frobenius morphism on $G$ is the application $F : G \to G$ defined by

$$F \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a^p & b^p \\ c^p & d^p \end{array} \right).$$

The reader can check that $F$ is a well-defined morphism of groups. The reason for introducing this particular morphism of groups is that it allows us to construct new $KG$-modules from the irreducible ones we already know at this point, i.e. the $S^rE$‘s, where $0 \leq r \leq p - 1$. Indeed, let $V$ be a $KG$-module on which $G$ acts according to the representation $\rho : G \to GL(V)$.

**Definition 2.3.1**

The Frobenius twist of $V$, denoted $V^F$, is the $KG$-module consisting in the same underlying $K$-vector space but on which $G$ acts according to the representation $\rho^F = \rho \circ F$.

As an example, consider $G = SL_2(9)$. By Theorem 2.1.5, there are at most nine non-isomorphic irreducible $KG$-modules and according to Theorem 2.2.4, $K, E$ and $S^2E$ are three of them. Moreover, since the Frobenius morphism is surjective, the Frobenius twist of any irreducible $KG$-module is again irreducible and so one can show directly that $E^F$ and $(S^2E)^F$ are two new irreducible $KG$-modules. However, four irreducible $KG$-modules may be missing.

2.3.2 A complete set of irreducibles

As we saw in the previous subsection, we need a few more tools if we want to give a complete description of the irreducible $KG$-modules. Let then $0 \leq r \leq q - 1$ be a non-negative integer and observe that $r$ can be written in a unique way as $r = a_0 + a_1p + a_2p^2 + \ldots + a_{n-1}p^{n-1}$, where $0 \leq a_0, \ldots, a_{n-1} \leq p - 1$ (such a writing is called the $p$-adic expansion of $r$). We then define the new $KG$-module

$$L(r) = S^{a_0}E \otimes (S^{a_1}E)^F \otimes \cdots \otimes (S^{a_{n-1}}E)^{F^{n-1}}. \quad (2.5)$$
Remark 2.3.1
For every $0 \leq r \leq q - 1$, the $KG$-module $L(r)$ defined in Equation (2.5) can be viewed as a submodule of $S'E$ using the injective morphism of $KG$-modules $\phi : L(r) \to S'E$ defined on generators by

$$\phi(u_0 \otimes u_1 \otimes \cdots \otimes u_{n-1}) = u_0u_1^r \cdots u_{n-1}^{p^n-1}.$$ 

In order to check that the map $\phi$ defined in Remark 2.3.1 is an injective morphism of $KG$-modules, the following statement is necessary.

Lemma 2.3.2 (Lucas Formula)
Let $s, r \in \mathbb{N}$ be two non-negative integers with $s \leq r$ having $p$-adic expansions $r = a_0 + a_1p + \cdots + a_mp^n$ and $s = b_0 + b_1p + \cdots + b_mp^n$. Then we have

$$C^r_s = \begin{cases} C^{a_0}_{b_0}C^{a_1}_{b_1} \cdots C^{a_m}_{b_m} & \text{if } b_i \leq a_i \text{ for every } 0 \leq i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the field $K$ has characteristic $p$, we have the polynomial equality

$$(x + 1)^r = (x + 1)^s (x^p + 1)^{a_1} \cdots (x^{p^m} + 1)^{a_m}$$

and then applying the well-known Binomial theorem to both sides of the equation yields

$$\sum_{i=0}^{r} C^r_i x^i = \left( \sum_{j_0=0}^{a_0} C^{a_0}_{j_0} x^{j_0} \right) \left( \sum_{j_1=0}^{a_1} C^{a_1}_{j_1} x^{j_1p} \right) \cdots \left( \sum_{j_m=0}^{a_m} C^{a_m}_{j_m} x^{j_mp^m} \right).$$

We now proceed exactly as we did in the previous subsection, ie. we first prove that for $0 \leq r \leq q - 1$ fixed, the $KG$-module $L(r)$ is generated by $e_1^r$ as a $KU^-$-module and then that $Ke_1^r$ is the unique $U$-invariant line in $L(r)$.

Lemma 2.3.3
The $KG$-module $L(r)$ is generated by $e_1^r$ as a $KU^-$-module.

Proof. Let $r = a_0 + a_1p + \cdots + a_{n-1}p^{n-1}$ be the $p$-adic expansion of $r$. We can then see that a $K$-basis of $L(r)$ is given by the vectors

$$e_1^{b_0+i_1p+\cdots+i_{n-1}p^{n-1}}, e_2^{r-(b_0+i_1p+\cdots+i_{n-1}p^{n-1})},$$

where $0 \leq i_\mu \leq a_\mu$ for every $0 \leq \mu \leq n - 1$. By Lemma 2.3.2, we can then affirm that the vector $e_1^r e_2^{r-1}$ ($0 \leq i \leq r$) belongs to this basis if and only if the binomial coefficient $C_i^r$ is non-zero. Now for every $a \in \mathbb{F}_q$, we have

$$u^{-}(a)e_1^r = \sum_{i=0}^{r} C_i^r a^{r-i} e_1^i e_2^{r-1}$$

and thus applying Lemma 2.2.1 yields the desired result.

\[\square\]
Lemma 2.3.4
The line $Ke_1^r$ is the unique $U$-invariant line in $L(r)$.

Proof. Similar to the proof of Lemma 2.2.3. □

We are now able to prove the main result of this chapter, which provides a complete description of the irreducible $KG$-modules.

Theorem 2.3.5
For every $0 \leq r \leq q - 1$, the $KG$-module $L(r)$ is irreducible. Moreover, the list $\{L(r)\}_{r=0,\ldots,q-1}$ forms a complete set of non-isomorphic irreducible $KG$-modules.

Proof. The first statement can be shown following exactly the same method as we did in the proof of Theorem 2.2.4, using Lemmas 2.3.3 and 2.3.4 instead of Lemmas 2.2.2 and 2.2.3 respectively.

Now we cannot use a simple dimension argument to see that $L(r)$ and $L(r')$ are non-isomorphic whenever $r \neq r'$ and so we shall use an argument on highest weights and highest weight vectors instead. By Lemma 2.1.3, the highest weight vector of an irreducible $KG$-module $V$ is uniquely defined (up to scalar multiplication) to be the generator (as a $K$-space) of the unique $U$-invariant line in $V$. Hence by Lemma 2.3.4, the element $v = e_1^r$ is a highest weight vector of $L(r)$ and its corresponding weight $(\lambda, \mu)$ is given as $(\lambda : h(t) \rightarrow t^r, \mu)$, where

$$\mu = \begin{cases} 0 & : \text{if } r \in \{0, \ldots, q-2\}, \\ -1 & : \text{if } r = q - 1. \end{cases}$$

To see this, use the well-known fact that the multiplicative group $\mathbb{F}_q^*$ is cyclic of order $q - 1$ together with the formula $(1 - x^n) = (1 + x + \ldots + x^{n-1})(1 - x)$. Therefore by Lemma 2.1.4, the $KG$-modules $L(r)$ and $L(r')$ are non-isomorphic whenever $r \neq r'$ and thus we conclude using Theorem 2.1.5. □

2.3.3 A special family of $KSL_2(q)$-modules

In the previous subsection, Theorem 2.3.5 gave us a classification of the irreducible $KG$-modules in terms of highest weights and we saw that in general, the $KG$-module $S^rE$ is not irreducible if $r \geq p$. Indeed, it contains the proper irreducible $KG$-module $L(r)$. Nevertheless, consider the special case $r = p^m - 1$, for some $1 \leq m \leq n$. Then $r = p-1+(p-1)p+\ldots+(p-1)p^{m-1}$ is the $p$-adic expansion of $r$ and the injective morphism of $KG$-modules $\phi : L(r) \rightarrow S^rE$ defined in Remark 2.3.1 becomes an isomorphism, since $\dim(L(r)) = p^m = \dim(S^rE)$.

Definition 2.3.2
A $KG$-module of the form $S^rE$ with $r$ as above is called a Steinberg module.

As an example, consider $G = SL_2(9)$. The Steinberg modules are $L(2) = S^2E$ and $L(8) = S^2E \oplus S^2E \cong S^8E$. 

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Let $K$ be an algebraically closed field of characteristic zero and let $n \in \mathbb{N}^*$ be a positive integer. Then the special linear Lie algebra $\mathfrak{sl}_n(K)$, defined by $\mathfrak{sl}_n(K) = \{g \in M_{n \times n} : \text{Tr}(g) = 0\}$, has dimension $n^2 - 1$ and standard basis

$\{e_{ij} : 1 \leq i \neq j \leq n\} \cup \{e_{kk} - e_{nn} : 1 \leq k \leq n - 1\},$

where $(e_{ij})_{\mu \nu} = \delta_{\mu i} \delta_{\nu j}$. Now in the case $n = 2$, we let $x$, $y$ and $h$ respectively denote the elements $e_{12}$, $e_{21}$ and $e_{11} - e_{22}$, ie.

$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

Furthermore, the reader can check that the latter satisfy the relations

$[h, x] = 2x, \quad [h, y] = -2y \quad \text{and} \quad [x, y] = h.$

The aim of this chapter is to study the representations of $\mathfrak{sl}_2(K)$ over $K$. More precisely, we shall give a complete set of irreducible $\mathfrak{sl}_2(K)$-modules, mention Weyl’s Theorem of complete reducibility and study how to decompose the tensor product of two irreducible $\mathfrak{sl}_2(K)$-modules. We principally follow the ideas of [Hum78] and [Maz10].

### 3.1 Preliminaries

We start by recalling some elementary background on representation theory of Lie algebras and explore some basic properties of $\mathfrak{sl}_2(K)$ in particular. Most of the results are still valid in the more general context of semisimple Lie algebras and we refer the reader to [Hum78] if interested in seeing more details.
3.1.1 Representations of Lie algebras

Let \( \mathfrak{L} \) be a Lie algebra over a field \( K \). Then an \( \mathfrak{L} \)-module is a \( K \)-vector space \( V \) together with a bilinear action \((\cdot, \cdot): \mathfrak{L} \times V \to V \) satisfying

\[
([x, y], v) = (x, (y, v)) - (y, (x, v)),
\]

for every \( x, y \in \mathfrak{L} \) and \( v \in V \). Throughout this chapter, we shall only work with finite-dimensional \( \mathfrak{L} \)-modules, unless otherwise indicated. As in group representation theory, one can then define a representation of \( \mathfrak{L} \) to be a morphism of Lie algebras \( \rho: \mathfrak{L} \to \mathfrak{gl}(V) \), where \( V \) is a \( K \)-vector space, and see that there is a one-to-one correspondence between \( \mathfrak{L} \)-modules and representations of \( \mathfrak{L} \).

A \( \mathfrak{L} \)-module is said to be irreducible if it has no non-zero proper submodules and analogously a representation of \( \mathfrak{L} \) is said to be irreducible if the corresponding \( \mathfrak{L} \)-module is irreducible. Finally, an \( \mathfrak{L} \)-module is said to be completely reducible if it is isomorphic to a direct sum of irreducible \( \mathfrak{L} \)-modules and a representation of \( \mathfrak{L} \) is called completely reducible if the corresponding \( \mathfrak{L} \)-module is completely reducible. The following result corresponds to Lemma 1.2.4 in the representation theory of groups. Its proof is exactly the same and so we refer the interested reader to the previously given reference.

**Lemma 3.1.1**

A finite-dimensional \( \mathfrak{L} \)-module is completely reducible if and only if every \( \mathfrak{L} \)-submodule is a direct summand.

3.1.2 Weights and weight spaces

Let \( K \) be an algebraically closed field, \( \mathfrak{L} \) be a nilpotent Lie algebra over \( K \) and let \( \rho: \mathfrak{L} \to \mathfrak{gl}(V) \) and \( \lambda: \mathfrak{L} \to K \) be two representations. Then \( \lambda \) is said to be a weight of \( \rho \) if there exists \( 0 \neq v \in V \) such that \((\rho(x) - \lambda(x)I)d_{v} = 0\) for every \( x \in \mathfrak{L} \). The \( K \)-space \( V_{\lambda} = \{ v \in V : \forall x \in \mathfrak{L}, \exists N_x \text{ s.t. } (\rho(x) - \lambda(x)I)d_{v}^{N_x}v = 0 \} \) is called the weight space of weight \( \lambda \).

**Theorem 3.1.2**

Let \( \mathfrak{L} \) be a nilpotent Lie algebra over \( K \) and \( V \) an \( \mathfrak{L} \)-module. Then \( V \) admits a decomposition (as an \( \mathfrak{L} \)-module) into a direct sum of weight spaces.

By Theorem 3.1.2, the Cartan subalgebra \( \mathfrak{h} = K h \) of \( \mathfrak{sl}_2(K) \) acts diagonally on any \( \mathfrak{L} \)-module \( V \), that is,

\[
V = \bigoplus_{\lambda \in K} V_{\lambda},
\]

where \( V_{\lambda} = \{ v \in V : h \cdot v = \lambda v \text{ for every } h \in \mathfrak{h} \} \) is the eigenspace associated to the eigenvalue \( \lambda \in K \). Since \( V \) is a finite-dimensional \( \mathfrak{sl}_2(K) \)-module, there are finitely many \( \lambda \) for which \( V_{\lambda} \) is different from zero, that is, a finite number of weights and weight spaces. Furthermore, the reader can easily check that if \( v \in V_{\lambda} \), then \( xv \in V_{\lambda+2} \) and \( yv \in V_{\lambda-2} \). Hence, there exists at least one weight \( \lambda \in K \) such that \( V_{\lambda+2} = 0 \) and any non-zero vector \( v \in V_{\lambda} \) is called a maximal vector of weight \( \lambda \). For such a fixed maximal vector \( v_0 \in V_{\lambda} \), we set \( v_{-1} = 0 \) and \( v_{k} = \frac{1}{k!}y^{k}v_{0} \), for \( k > 0 \).
Lemma 3.1.3
The following assertions hold:

(a) \( hv_k = (\lambda - 2k)v_k \);
(b) \( yv_k = (k + 1)v_{k+1} \);
(c) \( xv_k = (\lambda - k + 1)v_{k-1} \).

Moreover, if \( r \in \mathbb{N} \) denotes the smallest integer for which \( v_r \neq 0, v_{r+1} = 0 \), then the \( K \)-subspace \( V(r) \) of \( V \) spanned by the vectors \( v_0, \ldots, v_r \) is an irreducible \( \mathfrak{sl}_2(K) \)-submodule of \( V \).

Proof. The fact that formulas (a), (b) and (c) hold follows from straightforward calculations and thus its proof is left to the reader, who can consult [Hum78, Lemma 7.2 pp. 32] for details. Also, it is very easy to see that the \( K \)-vector space \( V(r) \) is an \( \mathfrak{sl}_2(K) \)-submodule of \( V \).

Finally, let \( W \) be a non-zero \( \mathfrak{sl}_2(K) \)-submodule of \( V(r) \) and pick \( 0 \neq w \in W \), which can be written as

\[
    w = \sum_{k=0}^{r} \alpha_k v_k,
\]

where \( \alpha_0, \ldots, \alpha_r \in K \). Then by applying \( x \) and \( y \) recursively, one can show that \( v_k \in W \) for every \( 0 \leq k \leq r \) and hence \( W = V(r) \), which leads to the desired result.

\( \square \)

3.2 Irreducible \( \mathfrak{sl}_2(K) \)-modules

In this section, we first give a description of the irreducible \( \mathfrak{sl}_2(K) \)-modules. Then we recall Weyl’s Theorem, which states that every \( \mathfrak{sl}_2(K) \)-module is completely reducible. Finally, we study the decomposition of the tensor product of any two irreducible \( \mathfrak{sl}_2(K) \)-modules, via the Clebsch-Gordan formula.

3.2.1 A complete set of irreducibles

Let \( E = E(2) \) denote the two-dimensional vector space over \( K \) having \( K \)-basis \( \{e_1, e_2\} \) (see Section 1.4) and extend the natural action of \( \mathfrak{sl}_2(K) \) on \( E \) to the symmetric algebra \( S(E) \) as follows: for \( x \in \mathfrak{sl}_2(K) \) and \( f, g \in S(E) \), let

\[
    x \cdot (fg) = (xf)g + f(xg).
\]

Then provided with this action, \( S(E) \) becomes an \( \mathfrak{sl}_2(K) \)-module. Moreover, the reader can check that for every \( r \in \mathbb{N} \), the \( r \)th symmetric power \( S^r E \) of \( E \) is an \( \mathfrak{sl}_2(K) \)-submodule of \( S(E) \).

Theorem 3.2.1
For every \( r \in \mathbb{N} \), the \( \mathfrak{sl}_2(K) \)-module \( S^r E \) is irreducible. Moreover, the list \( \{S^r E\}_{r \in \mathbb{N}} \) forms a complete set of non-isomorphic irreducible \( \mathfrak{sl}_2(K) \)-modules.
Proof. Let \( r \in \mathbb{N} \) be fixed. Then there is an isomorphism of \( \mathfrak{sl}_2(K) \)-modules \( \phi : V(r) \to S^r E \), which sends \( v_k \) to \( e_1^{-k}e_2^k \) for every \( 0 \leq k \leq r \). Therefore, \( S^r E \) is an irreducible \( \mathfrak{sl}_2(K) \)-module. On the other hand, by Lemma 3.1.3, any irreducible \( \mathfrak{sl}_2(K) \)-module is isomorphic to \( V(r) \) for some \( r \in \mathbb{N} \), which concludes the proof. 

\[ \square \]

Remark 3.2.2

Let \( V \) be an irreducible \( \mathfrak{sl}_2(K) \)-module and consider any maximal vector \( v_0 \in V \) of weight \( \lambda \in K \). Setting \( v_{-1} = 0 \) and \( v_k = \frac{1}{k!}y^k v_0 \), for \( k > 0 \) as before, Lemma 3.1.3 yields \( V \cong V(r) \), where \( r \) denotes the smallest integer for which \( v_r \neq 0 \). Then \( v_{r+1} = 0 \). Now applying formula (c) of the same Lemma to \( k = r + 1 \), we get \( 0 = (\lambda - r)v_r \), and hence \( \lambda = r \in \mathbb{N} \). In other words, the weight of a maximal vector is a non-negative integer, called the highest weight of \( V \). Consequently, Theorem 3.2.1 gave us a classification of the irreducibles in terms of highest weights.

3.2.2 Complete reducibility and the Clebsch-Gordan formula

Let \( r \in \mathbb{N} \) be a non-negative integer and suppose that \( \text{char}(K) = 0 \). From now on, we denote by \( V(r) \) the irreducible \( \mathfrak{sl}_2(K) \)-module of highest weight \( r \). Also recall the following result, which in fact holds for any semisimple Lie algebra over an algebraically closed field. Its proof can be found in [Hum78, VI.6.3, pp. 28-29].

Theorem 3.2.3 (Weyl’s Theorem)

Any \( \mathfrak{sl}_2(K) \)-module is completely reducible.

Now let \( V \) and \( W \) be two finite-dimensional \( \Sigma \)-modules with basis \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_m\} \) respectively. Then their tensor product \( V \otimes W \) can be provided with the structure of an \( \Sigma \)-module, defining the action on its basis by \( x \cdot (v_i \otimes w_j) = (xv_i) \otimes w_j + v_i \otimes (xw_j) \), for any \( x \in \Sigma \), any \( 1 \leq i \leq n \) and any \( 1 \leq j \leq m \).

Theorem 3.2.4 (Clebsch-Gordan formula)

Let \( s, r \in \mathbb{N} \) be two non-negative integers, with \( s \leq r \). Then we have

\[
V(r) \otimes V(s) \cong \bigoplus_{i=0}^{s} V(r+s-2i).
\]

Proof. We proceed using induction on \( s \). For \( s = 0 \), the result is obviously true. Now for \( s = 1 \), recall that the action of \( \mathfrak{sl}_2(K) \) on \( V(1) = E \) is given as \( xe_1 = 0 \), \( xe_2 = e_1 \), \( ye_1 = e_2 \), \( ye_2 = 0 \), \( he_1 = e_1 \), and \( he_2 = -e_2 \).

Now let \( v \) be a maximal vector of \( V(r) \) and consider \( v' = v \otimes e_1 \). Then the reader can verify that \( xv' = 0 \), \( hv' = (r+1)v \) and so \( V(r+1) \) is an irreducible submodule of \( V(r) \otimes V(1) \). Finally, Theorem 3.2.3 makes \( V(r+1) \) a direct summand.
The same argument applied to \( v'' = (yv) \otimes e_1 - rv \otimes e_2 \) gives us another direct summand \( V(r - 1) \) and a simple dimension argument yields

\[
V(r) \otimes V(1) \cong V(r + 1) \oplus V(r - 1).
\]

Finally, assume the result true for \( 0 \leq s \leq k - 1 \). Then

\[
(V(r) \otimes V(k - 1)) \otimes V(1) = \bigoplus_{i=0}^{r+k-1} V(r + k - 1 - 2i) \otimes V(1)
\]

\[
= \bigoplus_{i=0}^{r+k-1} \left( V(r + k - 1 - 2i) \otimes V(1) \right)
\]

\[
= \bigoplus_{i=0}^{r+k-1} \left( V(r + k - 2i) \oplus V(r + k - 2 - 2i) \right).
\]

On the other hand,

\[
V(r) \otimes (V(k - 1) \otimes V(1)) = V(r) \otimes V(k) \oplus V(k - 2) \otimes V(r)
\]

\[
= V(r) \otimes V(k) \oplus \left( \bigoplus_{i=0}^{r+k-2} V(r + k - 2 - 2i) \right)
\]

and thus we conclude using the associativity of the tensor product. \( \square \)
In this chapter, we let $K$ be an algebraically closed field and study the rational representations of $\text{SL}_2(K)$ over $K$. We start by introducing some elementary notions from algebraic geometry and then explore some basic properties of $\text{SL}_2(K)$ as an algebraic group. We then give a complete description of the irreducible $K\text{SL}_2(K)$-modules, depending on the characteristic of $K$ and observe in addition that if the latter is zero, every $K\text{SL}_2(K)$-module is completely reducible and we can easily decompose the tensor product of any two irreducible $K\text{SL}_2(K)$-modules. In fact, the representation theory of $\text{SL}_2(K)$ over $K$ is similar to the representation theory of the Lie algebra $\mathfrak{sl}_2(K)$. On the other hand, if the characteristic of $K$ is positive, not every $K\text{SL}_2(K)$-module is completely reducible and decomposing tensor products becomes far more complicated.

4.1 Some algebraic geometry

The aim of this first section is to familiarize the reader with basic concepts in algebraic geometry such as affine varieties, the Zariski topology, algebraic groups, rational representations and connectedness. We shall not give every detail in our construction, but the reader is welcome to consult [Bor91], [Hum75] or [Spr81], which are the classical references.

4.1.1 Affine varieties

Let $V$ be a set. Then the set $K^V$ of functions from $V$ to $K$ is obviously a commutative unital $K$-algebra, having addition and multiplication defined pointwise and unit $1 : g \mapsto 1_K$. Given a subalgebra $A \subset K^V$ and $v \in V$, we define the evaluation map $\epsilon_v \in \text{Hom}_{K\text{-alg}}(A,K)$ at $v$ by

\[
\epsilon_v : A \longrightarrow K
\]

\[
f \longmapsto f(v).
\]
Definition 4.1.1
An affine variety over \( K \) is a pair \((V,A)\) consisting of a set \( V \) and a finitely generated \( K \)-subalgebra \( A \) of \( K^V \) such that the map
\[
\epsilon : V \rightarrow \text{Hom}_{K-\text{alg}}(A,K)
\]
\[
v \mapsto \epsilon_v
\]
is a bijection. We write \( K[V] \) for \( A \) and call it the coordinate algebra of \( V \).

As an example, consider \( V = K^n \) and let \( A = K[X_1,\ldots,X_n] \) denote the \( K \)-algebra generated by the coordinate functions \( X_i \in K^V \), which are given as \( X_i(v) = v_i \) for \( v = (v_1,\ldots,v_n) \in V \). By definition, the \( K \)-algebra \( A \) is finitely generated. Furthermore,
\[
\epsilon_v = \epsilon_w \iff \epsilon_v(f) = \epsilon_w(f) \text{ for every } f \in A
\]
\[
\iff f(v) = f(w) \text{ for every } f \in A
\]
\[
\iff X_i(v) = X_i(w) \text{ for every } 0 \leq i \leq n
\]
\[
\iff v_i = w_i \text{ for every } 0 \leq i \leq n
\]
and hence the map \( \epsilon : V \rightarrow \text{Hom}_{K-\text{alg}}(A,K) \) is injective. Finally, if \( \theta \) is a morphism of \( K \)-algebras from \( A \) to \( K \), then \( \theta \) is easily seen to be the image of \( v = (\theta(X_1),\ldots,\theta(X_n)) \) under \( \epsilon \). Consequently, the latter is surjective as well and \((V,A)\) becomes an affine variety over \( K \), which is usually written \( k^n \) and called the affine \( n \)-space over \( K \).

Definition 4.1.2
A map \( \phi : V \rightarrow W \) between two affine varieties is called a morphism of affine varieties if \( g \circ \phi \in K[V] \) for every \( g \in K[W] \). This gives rise to a new map \( \phi^# : K[W] \rightarrow K[V] \) such that \( \phi^#(g) = g \circ \phi \), for \( g \in K[W] \), which we call the comorphism corresponding to \( \phi \).

Let \((V,A)\) be an affine variety over \( K \), let \( S \subset A \) be a subset of \( A \) and denote by \( \mathcal{V}(S) = \{ v \in V : f(v) = 0 \ \forall f \in S \} \) the set constituted of those \( v \in V \) which are annihilated by every \( f \in S \). Then one can show that the sets \( \mathcal{V}(S), S \subset A \) form the closed sets of a topology on \( V \), known as the Zariski topology.

Lemma 4.1.1
Any closed subset of \( V \) is an affine variety in its own rights. In particular, any subset of affine \( n \)-space specified by polynomial equations is an affine variety.

Proof. Let \( W = \mathcal{V}(S) \) be a closed subset of \( V \) and consider the \( K \)-algebra \( B = \{ f | W : f \in A \} \). Clearly \( B \) is finitely generated and if \( \epsilon_w : B \rightarrow K \) denotes the restriction of the evaluation map \( \epsilon_w \) at \( w \in W \), then obviously \( w_1 = w_2 \) whenever \( \epsilon_{w_1} = \epsilon_{w_2} \). Moreover, if \( \theta \in \text{Hom}_{K-\text{alg}}(B,K) \), then the map \( \theta : A \rightarrow K \) sending \( f \in A \) to \( \theta(f)|W \) is a morphism of \( K \)-algebras from \( A \) to \( K \). Hence there exists some \( v \in V \) such that \( \epsilon_v = \theta \). Now for every \( f \in S \), we have \( f(v) = \epsilon_v(f) = \theta(f) = \theta(f)|W = 0 \), thus \( v \in W \). Finally, for any \( g \in B \), we have \( \theta(g) = \theta(g)|W = \epsilon_v(g)|W = g|W(v) = g(v) = \epsilon'_v(g) \) and so the map \( \epsilon' : W \rightarrow \text{Hom}_{K-\text{alg}}(B,K) \) is a bijection as desired. \( \square \)
We now show a similar result concerning a special class of open sets. Let \((V, A)\) be an affine variety and let \(0 \neq f \in A\). The set \(V_f = \{v \in V : f(v) \neq 0\}\) is the complement of \(V(f)\) in \(V\) and thus is open in \(V\). Such sets are said to be principal open sets. Also define the \(K\)-algebra

\[ A_f = \left\{ \frac{a}{r^r} : a \in A, r \geq 0 \right\}. \]

**Lemma 4.1.2**

The pair \((V_f, A_f)\) is an affine variety. Hence any principal open set is an affine variety in its own rights.

**Proof.** Left to the reader. \(\square\)

**Examples:**

(a) Consider the set \(V = M_{n \times n}(K)\) of \(n \times n\) matrices with entries in \(K\). Then \(V\) can be viewed as the affine \(n\)-space over \(K\) with coordinate algebra \(K[V] = K[X_{11}, X_{12}, \ldots, X_{nn}]\), where \(X_{ij}(A) = A_{ij}\) for any \(A \in V\).

(b) Let \(V\) be as in (a) and let \(d : V \to K\) denote the determinant map. Then \(\text{SL}_n(K) = \mathcal{V}(d - 1)\) and thus is a closed subset of \(V\) and so is an affine variety by Lemma 4.1.1, with corresponding coordinate algebra \(K[\text{SL}_n(K)] = K[X'_{11}, \ldots, X'_{nn}]\), where \(X'_{ij}\) denotes the restriction of \(X_{ij}\) to \(\text{SL}_n(K)\).

(c) Let \(V\) and \(d\) be as in (b). Then \(\text{GL}_n(K) = V_d\) and thus is a principal open set of \(V\) and so is a affine variety, by Lemma 4.1.2, with corresponding coordinate algebra \(K[V_d] = K[X_{11}, X_{12}, \ldots, X_{nn}]\).

Finally, consider two affine varieties \(V, W\) over \(K\). We have a natural morphism of \(K\)-algebras \(i : K[V] \otimes K[W] \to K^{V \times W}\) defined on generators by

\[ i(f \otimes g)(v, w) = f(v)g(w). \]

Since \(i\) is injective, we may view the tensor product \(K[V] \otimes K[W]\) as a \(K\)-subalgebra of \(K^{V \times W}\). Now the reader can check that \((V \times W, K[V] \otimes K[W])\) is an affine variety (see [Spr81, ð 1.5 pp. 13-14] for details) and so from now on, we write \(K[V \times W]\) for \(K[V] \otimes K[W]\).

### 4.1.2 Algebraic groups

An **algebraic group** is a group \(G\) which is also an affine variety and such that the multiplication map \(m : G \times G \to G\) and inversion map \(i : G \to G\) are morphisms of affine varieties. A map \(\phi : G \to H\) between two such groups is a **morphism of algebraic groups** if it is both a morphism of groups and of affine varieties.

**Examples:**

(a) Denote by \(G_a\) the additive group over \(K\). Then \(G_a\) is the affine 1-space over \(K\) and so has coordinate algebra \(K[G_a] = K[X]\), where \(X(\alpha) = \alpha\) for every \(\alpha \in K\). Moreover, it is easy to see that \(X \circ m = 1 \otimes X + X \otimes 1\) and so belongs to \(K[G_a] \otimes K[G_a] = K[G_a \times G_a]\). Similarly, \(X \circ i = -X \in K[G_a]\) and hence \(G_a\) is an algebraic group.
(b) Denote by $G_m$ the multiplicative group over $K^*$. Then $G_m$ is the principal open set $A^1_X$ and so has coordinate algebra $K[G_m] = K[X, X^{-1}]$, where $X(\alpha) = \alpha$ as before. Moreover, it is easy to see that $X \circ m = X \otimes X$, $X^{-1} \circ m = X^{-1} \otimes X^{-1}$ and $X \circ i = X^{-1}$. Therefore, $G_m$ is an algebraic group.

(c) Let $G = \text{GL}_n(K)$. Recall that $G$ is an affine variety over $K$ with corresponding coordinate algebra $K[G] = K[X_{11}, \ldots, X_{nn}]$, where $X_{ij}(g) = g_{ij}$ for every $g \in G$ and $d : G \to K^*$ is the determinant map. Now for any $0 \leq i, j \leq n$, one can easily show that

$$X_{ij} \circ m = \sum_{k=1}^{n} X_{ik} \otimes X_{kj} \in K[G] \otimes K[G] = K[G \times G]$$

and similarly,

$$d^{-1} \circ m = d^{-1} \otimes d^{-1} \in K[G] \otimes K[G] = K[G \times G].$$

Therefore, since $K[G]$ is generated as a $K$-algebra by the coordinate functions together with $d^{-1}$ and since the map $f \mapsto f \circ m$ is a morphism of $K$-algebras, the multiplication map $m$ is a morphism of affine varieties. A similar argument shows that the inversion map $i$ also is a morphism of affine varieties and thus $G$ is an algebraic group.

(d) Any closed subgroup of $\text{GL}_n(K)$ is an algebraic group. In particular, $\text{SL}_n(K)$ is an algebraic group, since it is the subgroup of $\text{GL}_n(K)$ constituted of those elements annihilated by the map $d - 1 \in K[\text{GL}_n(K)]$.

### 4.1.3 The class of rational modules

Let $G$ be a group, $V$ be a $KG$-module and $\rho : G \to \text{GL}(V)$ be the representation afforded by $V$. If $\{v_1, \ldots, v_n\}$ denotes a $K$-basis of $V$, we have

$$\rho(g)(v_i) = \sum_{j=1}^{n} \rho_{ji}(g)v_j,$$

this for any $g \in G$, $1 \leq i \leq n$. The functions $\rho_{ij} \in K[G]$ are called coefficient functions of the $KG$-module $V$. Also, the $K$-space

$$\text{cf}(V) = \langle \rho_{ij} : 1 \leq i, j \leq n \rangle_K$$

generated by the coefficient functions is called the coefficient space of $V$. The reader can check that $\text{cf}(V)$ is independent of the choice of a $K$-basis of $V$.

**Definition 4.1.3**

Let $G$ be an algebraic group and let $V$ be a $KG$-module. Then $V$ is said to be rational if its coefficient space lies in the coordinate algebra $K[G]$ of $G$.

The reader can verify that the class of rational $KG$-modules is closed under duality, tensor products, sums and subquotients. As a first example, consider $G = \text{GL}_n(K)$ and let $E = E(n)_K$ denote the $n$-dimensional space of column vectors over $K$. Then $E$ is clearly a rational $KG$-module and thus the symmetric and exterior algebras $S(E)$ and $\Lambda(E)$ are rational $KG$-modules.
Lemma 4.1.3
Let $V$ be a rational $K\mathbb{G}$-module, $W$ a $K$-subspace of $V$ and $v \in V$. Then the set $S = \{ g \in G : gv \in W \}$ is closed with respect to the Zariski topology on $G$.

Proof. Let $\rho : G \to \text{GL}(V)$ be the representation afforded by $V$ and fix a basis $\{v_1, \ldots, v_m\}$ of $W$, which we extend to a basis $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ of $V$. Then writing $v$ in this basis as $v = \sum_{i=1}^{n} \alpha_i v_i$, we see that $gv$ belongs to $W$ if and only if there exist $\lambda_1, \ldots, \lambda_m \in K$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \rho_{ji}(g)v_j - \sum_{k=1}^{m} \lambda_k v_k = 0.$$ 

Therefore the set $S = \{ g \in G : gv \in W \}$ corresponds to the set of zeros of functions lying in some subset of $K[G]$ and hence is closed in $G$.

\qed

4.1.4 Irreducibility and Connectedness

Let $X$ be a topological space. Then $X$ is said to be Noetherian if every ascending chain of open sets is stationary, or equivalently, if every descending chain of closed sets is stationary. Notice that any affine variety $V$ is Noetherian. Indeed, if $Z_1 \supset Z_2 \supset \ldots$ is a descending chain of closed subsets in $V$, then we get an ascending chain of ideals $\mathcal{N}(Z_1) \subset \mathcal{N}(Z_2) \subset \ldots$ in $K[V]$, where for $Z \subset V$, we set $\mathcal{N}(Z) = \{ f \in K[V] : f(z) = 0 \text{ for every } z \in Z \}$. Therefore, since $K[V]$ is a Noetherian ring by Hilbert’s basis theorem, we get the desired result. Also $X$ is said to be irreducible if it is impossible to write it as a union of two proper closed subsets, or equivalently, if any two non-empty open subsets never intersect trivially.

Lemma 4.1.4
An affine variety $V$ is irreducible if and only if its coordinate algebra $K[V]$ is an integral domain.

Proof. Let $f, g \in K[V]$ with $fg = 0$. Then $V = \mathcal{V}(fg) = \mathcal{V}(f) \cup \mathcal{V}(g)$. Consequently, if $V$ is irreducible, we must have $\mathcal{V}(f) = V$ or $\mathcal{V}(g) = V$. Therefore, either $f$ or $g$ is identically zero and thus $K[V]$ is an integral domain. Conversely, if $K[V]$ is an integral domain, then $fg = 0$ if and only if $f = 0$ or $g = 0$ and so either $\mathcal{V}(f) = V$ or $\mathcal{V}(g) = V$.

\qed

Let $X$ be a set and let $X_1, \ldots, X_n$ be subsets whose union is $X$. Then the expression $X = X_1 \cup \ldots \cup X_n$ is said to be irredundant if there are no inclusions among the $X_i$’s and we refer to [Spr81, Proposition 1.2.4 pp. 4-5] for a proof of the following statement.

Theorem 4.1.5
Let $X$ be a Noetherian topological space. Then $X$ can be written as an irredundant union of finitely many irreducible closed subsets $X = X_1 \cup \ldots \cup X_n$ in a unique way (up to order).
Now let $G$ be an algebraic group. By Theorem 4.1.5, $G = X_1 \cup \ldots \cup X_n$ for some irreducible closed subsets $X_1, \ldots, X_n$ of $G$ and without loss of generality, we can assume $1 \in X_1$. For $g \in G$, we have $G = gG = gX_1 \cup \ldots \cup gX_n$ and each $gX_i$ is a closed irreducible subset of $G$. Hence by Theorem 4.1.5, for every $1 \leq i \leq n$, there exists $1 \leq j \leq n$ such that $gX_i = X_j$.

**Lemma 4.1.6**  
*The irreducible components of $G$ are disjoint. Hence any connected algebraic group is irreducible.*

**Proof.** Assume for a contradiction that $x \in X_i \cap X_j$ for some $x \in G$ and $i \neq j$. Then for any $g \in G$, we have $g \in gx^{-1}X_1 \cap gx^{-1}X_j$, so every $g \in G$ belongs to two irreducible components. Consequently, $X_1 \subset X_2 \cup \ldots \cup X_n$ and $X_1 = (X_1 \cap X_2) \cup \ldots \cup (X_1 \cap X_n)$ which by irreducibility of $X_1$, gives $X_1 \cap X_i = X_1$ for some $i \neq 1$ and thus $X_1 \subset X_i$, contradicting Theorem 4.1.5.

By Lemma 4.1.6, $X_1$ is the unique irreducible component of $G$ containing 1. Hence for any $g \in X_1$ we have $g \cdot 1 \in X_1 \cap gX_1$ and thus $gX_1 = X_1$. We also have $g^{-1}X_1 = X_1$ and hence $g^{-1} \in X_1$. Therefore, $X_1$ is a closed subgroup of $G$, denoted $G^0$ and the cosets of $G^0$ are the irreducible components of $G$, ie.

$$G = G^0 \cup g_1G^0 \cup \ldots \cup g_nG^0$$

for some $g_1, \ldots, g_n \in G$. Since there are finitely many of them, $G^0$ has finite index in $G$ and since $G^0$ is irreducible, it is in particular connected.

**Lemma 4.1.7**  
*Let $H$ and $K$ be two closed connected subgroups of $G$ and assume in addition that $K$ has finite index in $G$. Then $H \subset K$. In particular, the irreducible component $G^0$ of $G$ is the unique closed connected subgroup of $G$ of finite index.*

**Proof.** Since $K$ has finite index in $G$, there exist $g_1, \ldots, g_n \in G$ such that $G = K \cup g_1K \cup \ldots \cup g_nK$. Therefore, we can write $H$ as

$$H = (H \cap K) \cup (H \cap g_1K) \cup \ldots \cup (H \cap g_nK),$$

which is a union of finitely many closed subsets of $H$ and thus by connectedness of the latter, we have $H \cap K = H$ and $H \cap g_iK = \emptyset$ for every $1 \leq i \leq n$.

We now use Lemma 4.1.7 in order to show that the special linear group $\text{SL}_2(K)$ over $K$ is connected. In fact, the exact same argument works for $\text{SL}_n(K)$, $n \in \mathbb{N}$ arbitrary, by the well-known fact that it is generated by transvections (see Remark 1.1.2), and even for all Chevalley groups, since they also are generated by groups which are isomorphic to $G_2$.

**Corollary 4.1.8**  
*The group $\text{SL}_2(K)$ is a connected algebraic group. In particular, its coordinate algebra $K[\text{SL}_2(K)]$ is an integral domain.*
Proof. The unipotent subgroups $U$ and $U^{-1}$ of $\text{SL}_2(K)$ are closed and connected, since they are isomorphic to $\mathbb{G}_a$. Hence by Lemma 4.1.7, $U, U^{-1} \subset G^0$ and so Lemma 1.1.1 yields $\text{SL}_2(K) \subset G^0$. Finally, the second statement follows immediately from Lemma 4.1.4.

4.2 $\text{SL}_2(K)$ as a connected algebraic group

Let $K$ be an algebraically closed field and set $G = \text{SL}_2(K)$. The aim of this section is to introduce the prerequisites necessary to classify the irreducible rational $KG$-modules.

4.2.1 Weights and weight spaces

Let $V$ be a rational $KG$-module. The diagonal subgroup $T$ of $G$ obviously acts on $V$ and we will show that $V$ is completely reducible as a $KT$-module. This statement still holds if $G$ denotes any algebraic group and where we take $T$ to be a torus, i.e. a subgroup of $G$ isomorphic to some $\mathbb{D}_n$. We refer the reader to [Spr81, pp. 52-53] for a proof of the general result.

Lemma 4.2.1

Let $V$ be a one-dimensional rational $KT$-module. Then there exists $r \in \mathbb{Z}$ such that the action of $T$ on $V$ is given as

$$h(t)v = t^rv, \text{ for every } t \in K^*, v \in V.$$  

Proof. Let $\lambda : T \to K^*$ denote the representation afforded by $V$ and recall that the coordinate algebra $K[T]$ of $T$ is given as $K[T] = K[X_{11}, X_{11}^{-1}]$. Hence we have

$$\lambda(h(t)) = P(t), \quad t \in K^*,$$

where $P \in K[x, x^{-1}]$ is a Laurent polynomial in the indeterminate $x$. Now using the fact that $\lambda$ is a representation, one can show that $P$ has to be of the form $P(x) = x^r$ for some $r \in \mathbb{Z}$ and so the statement holds.

Theorem 4.2.2

Every rational $KT$-module is completely reducible.

Proof. Let $V$ be a rational $KT$-module and let $\rho : T \to GL(V)$ denote the representation afforded by $V$. We proceed by induction on $\dim(V) = n$. If $V$ is one-dimensional, then $V$ is irreducible. If $n = 2$, $V$ is either irreducible or contains a one-dimensional $KT$-submodule $W$. In the latter case, Lemma 4.2.1 yields

$$\rho(h(t)) = \begin{pmatrix} t^r & P(t) \\ 0 & t^s \end{pmatrix}, \quad t \in K^*,$$

where $P \in K[x, x^{-1}]$ is a Laurent polynomial in the indeterminate $x$. Again, using the fact that $\rho$ is a representation, one can show that $P$ has to be identically zero and so we get the desired result for $n = 2$. 

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Now assume the Theorem true for \( n \geq 2 \), let \( V \) be an \((n + 1)\)-dimensional \( KT \)-module and consider a maximal \( KT \)-submodule \( W \) of \( V \). By Lemma 1.2.8, \( W \) has codimension one in \( V \) and is completely reducible by our induction hypothesis. Therefore there exist two proper \( KT \)-submodules \( W_1, W_2 \) of \( W \) such that \( W = W_1 \oplus W_2 \) and the morphism of \( KT \)-modules

\[
\phi : V \to V/W_1 \oplus V/W_2
\]

has Kernel \( \text{Ker}(\phi) = W_1 \cap W_2 = 0 \). Consequently, \( V \) can be viewed as a \( KT \)-submodule of the direct sum of two completely reducible \( KT \)-modules and so the result follows.

\[\square\]

Let \( V \) be a rational \( KG \)-module. By Theorem 4.2.2, \( V \) is isomorphic to a direct sum of irreducible \( KT \)-modules, which by Lemma 1.2.8 are one-dimensional. Thus using Lemma 4.2.1, we conclude that \( V \) has the grading as a \( KT \)-module

\[
V = \bigoplus_{r \in \mathbb{Z}} V_r,
\]

where \( V_r = \{ v \in V : h(t)v = t^rv \text{ for every } t \in K^* \} \). In this situation, \( r \in \mathbb{Z} \) is called a weight of \( V \) whenever \( V_r \) is non-zero. The latter is called the weight space associated to the weight \( r \). Finally, the greatest integer \( r \) such that \( V_r \) is non-zero is called the highest weight of \( V \). Again, this can be done in a more general context (see [Spr81, pp.187]).

**Lemma 4.2.3**

*For every \( r \in \mathbb{Z} \), the weight spaces \( V_r \) and \( V_{-r} \) are isomorphic. Hence the highest weight of a rational \( KG \)-module is a non-negative integer.*

**Proof.** Let \( g = h(t) \in T \). Then a simple calculation yields \( w^{-1}gw = g^{-1} \) and hence for \( v \in V_r \), we get \( g(ww^{-1}gv) = (ww^{-1}gv)v = (ww^{-1}g^{-1}v)v = w(g^{-1}v) = t^{-r}vw \). This gives rise to a morphism of \( KT \)-modules \( \theta : V_r \to V_{-r} \) sending \( v \in V_r \) to \( wv \in V_{-r} \), and the reader can check that \( \theta \) is an isomorphism.

\[\square\]

Now let \( 0 \to X \to V \to Y \to 0 \) be an exact sequence of rational \( KG \)-modules. Then the proof of the following result is left to the reader.

**Lemma 4.2.4**

*The naturally induced sequence of \( K \)-spaces \( 0 \to X_r \to V_r \to Y_r \to 0 \) is exact. In particular, for any two rational \( KG \)-modules \( V, W \) and any \( r \in \mathbb{Z} \), we have \( (V \oplus W)_r = V_r \oplus W_r \).*

Finally, let \( V, W \) be two rational \( KG \)-modules. Then the reader can check that the following assertion holds.

**Lemma 4.2.5**

*For any \( r \in \mathbb{Z} \), we have \( (V \otimes W)_r = \bigoplus V_i \otimes W_j \), where the sum is over all \( i, j \in \mathbb{Z} \) satisfying \( i + j = r \).*
4.2.2 The formal character

Let $V$ be a rational $KG$-module. Then the formal character of $V$ is the Laurent polynomial $\Phi_V \in \mathbb{Z}[x,x^{-1}]$ defined by

$$\Phi_V = \sum_{r \in \mathbb{Z}} \dim(V_r)x^r.$$ 

By Lemma 4.2.3, $\Phi_V$ is a symmetric Laurent polynomial. Moreover, it does not depend on the characteristic of the field $K$, which is an obvious but very important property.

Examples:

(a) The formal character of the trivial $KG$-module $K$ is $\Phi_K = 1$.

(b) The weights of the $KG$-module $E = E_2(K) = \langle e_1, e_2 \rangle_K$ are $-1, 1$ and then the formal character of $E$ is $\Phi_E = x + x^{-1}$.

(c) More generally, let $r \in \mathbb{N}$ be a non-negative integer and consider the $r$th symmetric power $S^rE$ with its $KG$-module structure. The weights of $S^rE$ are $r, r-2, \ldots, -r+2, -r$ and then the formal character of $S^rE$ is

$$\Phi_{S^rE} = \sum_{i=0}^{r} x^{r-2i}.$$ 

In particular, we observe that the list $\{\Phi_{S^rE}\}_{r \in \mathbb{N}}$ is linearly independent and hence forms a basis of the space of symmetric Laurent polynomials in the indeterminates $x, x^{-1}$.

Lemma 4.2.6

Let $W$ be a submodule of a rational $KG$-module $V$. Then $\Phi_V = \Phi_W + \Phi_{V/W}$. In particular, the formal character of the direct sum of finitely many $KG$-modules equals the sum of the formal character of each direct summand.

Proof. This follows directly from Lemma 4.2.4. 

Lemma 4.2.7

Let $V$ and $W$ be two rational $KG$-modules. Then $\Phi_{V \otimes W} = \Phi_V \Phi_W$.

Proof. This follows directly from Lemma 4.2.5.

Now let $V$ be a rational $KG$-module and assume that $\{L(\lambda) : \lambda \in \Lambda\}$ forms a complete set of non-isomorphic irreducible rational $KG$-modules.

Theorem 4.2.8

The formal character $\Phi_V$ of $V$ is given as

$$\Phi_V = \sum_{\lambda \in \Lambda} [V : L(\lambda)]\Phi_{L(\lambda)}.$$ 

Proof. Consider a composition series $V \supset V_1 \supset \ldots \supset V_{r-1} \supset V_r = 0$ for $V$. Then using recursively Lemma 4.2.6 yields the desired result.
4.2.3 The unipotent subgroups $U$ and $U^-$

We now explore some very important properties verified by the two unipotent subgroups $U$ and $U^-$ of $G$. Let then $V$ be a rational $KG$-module, $r \in \mathbb{Z}$ be a weight of $V$ and consider $0 \neq v \in V_r$.

**Lemma 4.2.9**

For every $i, j \in \mathbb{Z}$, there exist $v_i \in V_{r+2i}$ and $w_j \in V_{r-2j}$ such that for every $a \in K$, we have

(a) $u(a)v = v_0 + av_1 + \ldots + a^k v_k$,

(b) $u^-(a)w = w_0 + aw_1 + \ldots + a^j w_l$.

**Proof.** The arguments are the same in the two cases and so we only give a proof of (a). Let $\{x_1, \ldots, x_n\}$ be a $K$-basis of $V$, with $x_1 = v$ and let $\rho : G \to \text{GL}(V)$ be the representation afforded by $V$. Then for any $a \in K$, we have

\[
u(a) = \sum_{k=1}^{n} \rho_k(u(a))x_k\]

where for every $1 \leq k \leq n$, $P_k \in K[x]$ denotes a polynomial. Thus there exist $v_0, \ldots, v_k \in V$ such that for every $a \in K$, we have

\[
u(a) = v_0 + av_1 + \ldots + a^k v_k.

Now for $t \in K^*$ fixed, an easy calculation yields $h(t)u(a)h(t^{-1}) = u(at^2)$ and hence

\[
h(t)u(a)h(t^{-1})v = v_0 + at^2 v_1 + \ldots + a^k t^{2k} v_k.

On the other hand,

\[
h(t)u(a)h(t^{-1})v = h(t)u(a)h^{-1}v = t^{-r} h(t) u(a) v = t^{-r} h(t)(v_0 + av_1 + \ldots + a^k v_k) = t^{-r} h(t)v_0 + at^{-r} h(t)v_1 + \ldots + a^k t^{-r} h(t)v_k

and thus for every $t \in K^*$, $a \in K$, we have

\[
(t^{-r} h(t) - 1)v_0 + (t^{-r} h(t) - t^2)av_1 + \ldots + (t^{-r} h(t) - t^{2k})a^k v_k = 0.

Therefore, for every $0 \leq i \leq k$, we have $(t^{-r} h(t) - t^{2i})v_i = 0$, which is the same as saying that $v_i \in V_{r+2i}$.

\[\square\]
Remark 4.2.10
Observe that if we replace $a$ by 0 in Lemma 4.2.9, we get $v_0 = v$ and $w_0 = w$. In particular, if $v$ is a highest weight vector of a $KG$-module $V$, then $U$ acts trivially on $Kv$.

Lemma 4.2.11
Every irreducible rational $KU$-module is trivial (the statement also holds for $U^-$ instead of $U$). Hence the only one-dimensional rational $KG$-module is the trivial one.

Proof. Since $U$ is abelian, every irreducible $KU$-module $V$ is one-dimensional, by Lemma 1.2.8. Now for any $a \in K$ and any $v \in V$, we have $u(a)v = P(a)v$, where $P \in K[x]$ is a polynomial in the indeterminate $x$, and since $V$ is a $KU$-module, we must have $P(a + b) = P(a)P(b)$ for every $a, b \in K$, which forces $P = 1$. Finally, the second affirmation follows immediately from Lemma 1.1.1.

4.2.4 A dense subset of $SL_2(K)$
Consider the subset $U^TU$ of $G$. Then the reader can easily check that an element $g \in G$ belongs to $U^TU$ if and only if $X_{11}(g)$ is non-zero. In other words, we have $U^TU = G_{X_{11}} = \{ g \in G : X_{11}(g) \neq 0 \}$, i.e. $U^TU$ is a principal open set. Hence $G = G_{X_{11}} \cup V(X_{11})$, a union of closed subsets of $G$, the second of which is proper; thus Lemma 4.1.8 yields the following statement.

Lemma 4.2.12
The principal open set $U^TU$ is dense in $G$.

Now let $V$ be a rational $KG$-module having highest weight $r \in \mathbb{N}$ and let $0 \neq v \in V_r$ be a vector of weight $r$.

Theorem 4.2.13
The $K$-vector space $KU^TUv$ is a $KG$-submodule of $V$.

Proof. Consider the subset $S = \{ g \in G : gv \in KU^TUv \}$ of $G$. Then $S$ obviously contains $U^TU$, which is dense in $G$ by Lemma 4.2.12. On the other hand, $S$ is closed by Lemma 4.1.3 and so $S = S \supset U^TU = G$, i.e. $KU^TUv$ is a $KG$-submodule of $V$.

4.3 Irreducible rational $KSL_2(K)$-modules
We are finally ready to give a complete set of irreducible $KG$-modules, whenever $K$ has characteristic zero or $p > 0$. The idea is to proceed in the same way as we did for $SL_2(q)$ (and $sl_2(K)$), that is, to give a classification in terms of highest weights. We first need a replacement for Lemma 2.1.4.
Lemma 4.3.1
Any two irreducible $KG$-modules having the same highest weight are isomorphic.

Proof. Consider two irreducible $KG$-modules $V_1$ and $V_2$ having the same highest weight $r \in \mathbb{N}$ and let $v_1$ and $v_2$ be highest weight vectors of $V_1$ and $V_2$ respectively. If $v = v_1 + v_2 \in V_1 \oplus V_2$, Lemmas 4.2.9 and 4.2.13 yield

$$KGv = KU^{-1}Tu v = KU^{-1}v = KUv = Kv \oplus \left( \bigoplus_{i=1}^{r} Kx_i \right),$$

where $x_i \in V_{r-2i}$ for every $1 \leq i \leq r$. Therefore, $v_2$ does not belong to $KGv$ and so the projection map $\pi_1 : KGv \to V_1$ is an injective morphism of $KG$-modules. Moreover, $\pi_1(v) = v_1 \in V_1$ and thus $\pi_1$ is surjective as well. The same argument works for the projection map $\pi_2 : KGv \to V_2$ and so

$$V_1 \xleftarrow{\cong} KGv \xrightarrow{\cong} V_2.$$
Theorem 4.3.4
For every \( r \in \mathbb{N} \), the KG-module \( V(r) \) is irreducible. Furthermore, the list \( \{V(r)\}_{r \in \mathbb{N}} \) forms a complete set of non-isomorphic irreducible rational KG-modules in characteristic zero.

Proof. Let \( r \in \mathbb{N} \) be fixed. In order to prove the first statement, we proceed in the exact same way as in the proof of Theorem 2.2.4, ie. we first show that there is a unique \( U \)-invariant line \( KU_v \) in \( V(r) \) and that \( KU^{-v} \cdot v = V(r) \). This argument works since Lemma 4.2.11 plays the same role as Lemma 1.2.9 (b) did in the proof of Theorem 2.2.4. Finally, it is easy to check that \( V(r) \) has highest weight \( r \) and then Lemma 4.3.1 yields the desired result.

Now assume that \( K \) has characteristic \( p > 0 \) and for \( r \in \mathbb{N} \), let \( L(r) \) denote the KG-module defined as in (2.5). Notice that \( L(r) \) is a rational KG-module, since it is a tensor product of rational KG-modules.

Theorem 4.3.5
For every \( r \in \mathbb{N} \), the KG-module \( L(r) \) is irreducible. Furthermore, the list \( \{L(r)\}_{r \in \mathbb{N}} \) forms a complete set of non-isomorphic irreducible rational KG-modules in positive characteristic.

Proof. The KG-module \( L(r) \) is irreducible as a \( K \text{SL}_2(p) \)-module by Theorem 2.3.5 and thus is irreducible as a KG-module as well, since \( \mathbb{F}_p \subset K \). Finally, it is easy to check that \( L(r) \) has highest weight \( r \) and then Lemma 4.3.1 yields the desired result.

4.4 Additional results in characteristic zero

In this section, we consider an algebraically closed field \( K \) of characteristic zero and study how rational KG-modules decompose as direct sums of indecomposable (even irreducible by the next result) rational KG-modules.

Theorem 4.4.1
Every rational KG-module is completely reducible.

Proof. Let \( V \) be a rational KG-module and first assume that it contains a KG-submodule \( W \) of codimension one. As in the proof of Theorem 3.2.3 (see [Hum78, 6.3, pp. 28-29]), we can assume \( W \) irreducible and so by Theorem 4.3.4, \( W = V(r) \) for some \( r \in \mathbb{N} \). Now by Lemma 4.2.11, the sequence \( 0 \to V(r) \to V \to K \to 0 \) is exact. Dualizing (modulo \( ^{-1} : KG \to KG \)) and using Lemma 4.3.3, we get the short exact of rational KG-modules

\[
0 \to M \to V^* \to V(r) \to 0,
\]

where \( M \) is a KG-submodule of \( V^* \) which is isomorphic to the trivial KG-module \( K \).
If \( r \) is an odd integer, let \( V' \) be the sum of those weight spaces in \( V^* \) having even weight and \( V'' \) the complement of \( V' \) in \( V^* \) as defined below:

\[
V' = \bigoplus_{i \in \mathbb{Z}} V_{2i}^* \quad \text{and} \quad V'' = \bigoplus_{i \in \mathbb{Z}} V_{2i+1}^*.
\]

It is clear that \( V^* \) is isomorphic to the direct sum of \( V' = V(r) \) and \( V'' = K \) as a \( K \)-vector space (even as a \( KT \)-module) and by Lemma 4.2.9, \( V' \) and \( V'' \) are stable under the action of \( U \) and \( U^- \) and hence by Lemma 1.1.1, under the action of \( G \). Therefore \( V^* \cong V' \oplus V'' = V(r) \oplus K \) as \( KG \)-modules.

On the other hand, if \( r \) is an even integer, take \( 0 \neq v \in V^*_r \). Then as in the proof of Lemma 4.3.1, we have

\[
KGv = Kv \oplus \left( \bigoplus_{i=0}^r Kx_i \right),
\]

where \( x_i \in V_{r-2i}^* \) for every \( 0 \leq i \leq r \). Therefore, \( KGv \) has a one-dimensional zero weight space \( N \). By exactness of (4.1), \( M \cap N = 0 \) and hence

\[
V^* \cong V(r) \oplus M.
\]

Now let \( W \) be an arbitrary rational \( KG \)-submodule of \( V \) and recall that the \( K \)-space \( \text{Hom}_K(V,W) \) has the structure of a \( KG \)-module, where for every \( x \in G, \ v \in V \) and \( f \in \text{Hom}_K(V,W) \), we put \( (x \cdot f)(v) = xf(x^{-1}v) \). Now define the sets \( \mathcal{V} = \{ f \in \text{Hom}_K(V,W) : f|_W = 0 \} \) and \( \mathcal{W} = \{ f \in \mathcal{V} : f|_W = 0 \} \), which are two \( KG \)-submodules of \( \text{Hom}_K(V,W) \) fitting in the short exact sequence \( 0 \to \mathcal{W} \to \mathcal{V} \to K \to 0 \) by Lemma 4.2.11. Indeed, \( \text{codim}(\mathcal{W}, \mathcal{V}) = 1 \).

According to the special case considered above, \( \mathcal{V} \) has a one-dimensional \( KG \)-submodule complementary to \( \mathcal{W} \), which is spanned by some \( f \in \text{Hom}_K(V,W) \). Without loss of generality, we can assume \( f|_W = 0 \). Now \( f \) is a morphism of \( KG \)-modules and thus Ker(\( f \)) is a \( KG \)-submodule of \( V \). One can then easily conclude that \( V = W \oplus \text{Ker}(f) \) and thus the proof is complete by Lemma 1.2.4.

We are now ready to study the decomposition of the tensor product of any two symmetric powers, which are, since \( K \) is a field of characteristic zero, exactly the irreducible rational \( KG \)-modules by Theorem 4.3.4.

**Theorem 4.4.2** (Clebsch-Gordan Formula)

Let \( s, r \in \mathbb{N} \) be two non-negative integers with \( s \leq r \). Then we have

\[
V(r) \otimes V(s) \cong \bigoplus_{i=0}^s V(r + s - 2i).
\]

**Proof.** We proceed in a completely different way than in the case of the Lie algebra. In fact, the introduction of the formal character earlier makes everything easy. Indeed, using Lemmas 4.2.6 and 4.2.7, one can show that the \( KG \)-modules \( V(r) \otimes V(s) \) and \( \bigoplus_{i=0}^s V(r + s - 2i) \) have same formal character and thus by Theorem 4.3.2, they have same composition factors including multiplicities (up to isomorphism). Finally, Theorem 4.4.1 concludes the proof.

\[ \square \]

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Decomposition in characteristic $p > 0$

In the previous chapter, Theorem 4.4.2 gave us a way to decompose the tensor product of any two symmetric powers into a direct sum of irreducible $KSL_2(K)$-modules, where $K$ denoted an algebraically closed field of characteristic zero. Now in this situation, the symmetric powers formed a complete set of non-isomorphic irreducible rational $KSL_2(K)$-modules by Theorem 4.3.4. Let then $K$ be an algebraically closed field of characteristic $p > 0$. We could wonder how to decompose the tensor product of two irreducible $KSL_2(K)$-modules as well as how to decompose the tensor product of two symmetric powers. The first problem is studied in [DH05] and [DD09]. The second is unsolved even in characteristic 2, but we shall see a way to tackle it.

Lemma 5.0.3

Let $V$ be an indecomposable $GL_n(K)$-module and let $W$ be a $K$-subspace of $V$. Then $W$ is a $KGL_n(K)$-submodule of $V$ if and only if it is a $KSL_n(K)$-submodule of $V$.

Proof. Let $Z$ denote the center of $GL_n(K)$ and let $z \in Z$ be fixed. Then as a $K$-space, $V$ can be written as

$$V = \bigoplus_{\lambda \in K} V^\lambda, \quad (5.1)$$

where $V^\lambda = \{v \in V : zv = \lambda v\}$. Now let $\lambda \in K$, $v \in V^\lambda$ and $g \in GL_n(K)$. Then we have $z(gv) = (zg)v = (gz)v = g(zv) = \lambda gv$ and thus (5.1) is a decomposition of $V$ as a $KGL_n(K)$-module. Since we assumed $V$ indecomposable, $V = V^\lambda$ for some $\lambda \in K$, i.e. $z$ acts by scalar multiplication on $V$. Finally, using the fact that $GL_n(K) = ZSL_n(K)$ yields the desired result.
Remark 5.0.4
A direct consequence of Lemma 5.0.3 is that a decomposition of $V$ in terms of indecomposable $K\text{SL}_n(K)$-submodules can be immediately deduced from a decomposition in terms of indecomposable $K\text{GL}_n(K)$-submodules. Now there is a real advantage in working with $K\text{GL}_n(K)$-modules instead of $K\text{SL}_n(K)$-modules and we shall clarify this point in the next pages.

5.1 The Schur algebra

From now on, we let $K$ be an algebraically closed field of characteristic $p > 0$ and we write $G = \text{GL}_n(K)$. We also denote by $A_K(n)$ the $K$-subalgebra of $K[G]$ generated by the coordinate functions $X_{ij}$, i.e.

$$A_K(n) = K[X_{11}, \ldots, X_{nn}].$$

Since the coordinate functions $X_{ij}$ are algebraically independent over $K$, $A_K(n)$ can be regarded as the algebra of all polynomials over $K$ in $n^2$ indeterminates $X_{ij}$. Finally, for $r \in \mathbb{N}$, we let $A_K(n,r)$ be the $K$-subspace of $A_K(n)$ consisting of the elements expressible as homogeneous polynomials of degree $r$ in the $X_{ij}$’s. Clearly we have a grading (as $K$-spaces)

$$A_K(n) = \bigoplus_{r \in \mathbb{N}} A_K(n,r).$$

(5.2)

5.1.1 The $K$-spaces $A_K(n)$ and $A_K(n,r)$

First recall that a $K$-coalgebra $A = (A, \Delta, \epsilon)$ is a $K$-space together with $K$-linear maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to K$ satisfying

$$(Id_A \otimes \Delta) \circ \Delta = (\Delta \otimes Id_A) \circ \Delta,$$

$$(Id_A \otimes \epsilon) \circ \Delta = (\epsilon \otimes Id_A) \circ \Delta = Id_A.$$  

The maps $\Delta$ and $\epsilon$ are respectively called comultiplication and counit maps. Also a $K$-bialgebra is a $K$-space $A$ which is both a unital associative $K$-algebra and a $K$-coalgebra. In addition, we require the counit and multiplication maps to be morphisms of $K$-algebras.

It easy to see that the decomposition (5.2) is not a $K$-algebra grading. Indeed, the $K$-spaces $A_K(n,r)$’s are not subalgebras of $A_K(n)$. Nevertheless, consider the maps $\Delta : A_K(n) \to A_K(n) \otimes A_K(n)$ and $\epsilon : A_K(n) \to K$ given on generators as

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$$

and $\epsilon(X_{ij}) = \delta_{ij}.$

Lemma 5.1.1
The maps $\Delta$ and $\epsilon$ defined above provide the $K$-algebra $A_K(n)$ with the structure of a $K$-bialgebra. Moreover, the $K$-subspaces $A_K(n,r)$’s are subcoalgebras of $A_K(n)$ and thus (5.2) is a grading as $K$-coalgebras.

Proof. The proof only consists of straightforward calculations and thus is left to the reader.
Let $A = (A, \Delta, \epsilon)$ be a $K$-coalgebra. Then the linear dual $A^* = \text{Hom}_K(A, K)$ of $A$ can be provided with the structure of a unital associative $K$-algebra, where the multiplication of any two $\alpha, \beta \in A^*$ is given as $(\alpha \cdot \beta)(v) = (\alpha \otimes \beta)\Delta(v)$, this for every $v \in A$, and the unit $1_{A^*} = \epsilon$.

**Definition 5.1.1**
The Schur algebra $S_K(n, r)$ is the linear dual of $A_K(n, r)$, provided with multiplication and unit as above.

Finally, $G$ acts as algebra automorphisms on $A_K(n, r)$ by $(g \cdot f)(h) = f(hg)$ and by $(f \cdot g)(h) = f(gh)$, $g \in G$, $f, h \in A_K(n, r)$, making it a $KG$-bimodule. Now the reader can check that (5.2) is a grading as $KG$-bimodules and that the action of any element $g \in G$ on the coordinate functions is given as

$$g \cdot X_{ij} = \sum_{k=1}^n g_{kj} X_{ik} \text{ and } X_{ij} \cdot g = \sum_{k=1}^n g_{ik} X_{kj}.$$ 

(5.3)

5.1.2 The category of polynomial $KGL_n(K)$-modules

Let $M_K(n)$ (resp. $M_K(n, r)$) denote the category of $KG$-modules $V$ whose coefficient space $\text{cf}(V)$ lies in $A_K(n)$ (resp. $A_K(n, r)$). It is possible to prove (see [Mar08] p. 2-3) that any indecomposable module $V \in M_K(n, r)$ is homogeneous, ie. $V \in M_K(n, r)$ for some $r \in \mathbb{N}$. Therefore, from now on, we shall consider such an $r$ to be fixed.

**Definition 5.1.2**
The objects in $M_K(n, r)$ are called polynomial $KG$-modules (of degree $r$) and the afforded representations are called polynomial representations (of degree $r$) of $G$ over $K$.

Given $f \in K^G$, there is a unique linear extension $f \in K^{KG}$ of $f$, which sends any $\kappa = \sum \alpha_g g \in KG$ to

$$f(\kappa) = \sum \alpha_g f(g),$$

(5.4)
called the evaluation of $f$ at $\kappa$. Now for $g \in G$, let $e_g$ denote the element of $S_K(n, r)$ such that $e_g(f) = f(g)$ for every $f \in A_K(n, r)$. This gives rise to a map $e : G \rightarrow S_K(n, r)$ and an easy calculation yields $e_g e_h = e_{gh}$ for every $g, h \in G$ and $e_{1_G} = \epsilon$. Hence if we extend linearly the map $e$, we get a morphism of $K$-algebras $e : KG \rightarrow S_K(n, r)$, which for any $\kappa \in KG$ corresponds to the evaluation at $\kappa$ seen in (5.4).

**Lemma 5.1.2**
The morphism of $K$-algebras $e$ defined above is surjective.

**Proof.** Suppose for a contradiction that $\text{Im}(e)$ is a proper subspace of $S_K(n, r)$. Hence there exists $0 \neq f \in A_K(n, r)$ such that for every $g \in G$, $e_g(f) = f(g) = 0$, a contradiction.\[\square\]
Lemma 5.1.3
An element \( f \in K^G \) belongs to \( A_K(n, r) \) if and only if \( f(\text{Ker}(e)) = 0 \).

Proof. Let \( f \in A_K(n, r) \) and consider \( \kappa \in \text{Ker}(e) \). Then \( f(\kappa) = e(\kappa)(f) = 0 \) and thus the first direction is proved. Conversely, let \( f \in K^G \) be such that \( f(\text{Ker}(e)) = 0 \). By applying Lemma 1.3.1 to Lemma 5.1.2, the sequence
\[
0 \rightarrow S_K(n, r)^* \rightarrow (KG)^* \rightarrow \text{Ker}(e)^* \rightarrow 0
\]
is a short exact sequence. Therefore, there exists some \( y \in S_K(n, r)^* \) such that \( y(e(\kappa)) = f(\kappa) \), for all \( \kappa \in KG \). Consequently, there exists some \( c \in A_K(n, r) \) satisfying \( y(\xi) = \xi(c) \), this for all \( \xi \in S_K(n, r) \). In particular, choosing \( \xi = e(\kappa) \) leads to the desired result.

\[\square\]

Lemma 5.1.4
Let \( V \) be a \( KG \)-module. Then \( V \) belongs to \( M_K(n, r) \) if and only if \( \text{Ker}(e)V = 0 \).

Proof. Let \( \{v_1, \ldots, v_n\} \) be a \( K \)-basis of \( V \) and denote by \( \rho_{ij} \) the matrix afforded by the action of \( KG \) on this basis. Then
\[
\text{Ker}(e)V = 0 \iff \rho_{ij}(\text{Ker}(e)) = 0 \forall 1 \leq i, j \leq n
\]
\[
\iff \rho_{ij} \in A_K(n, r) \forall 1 \leq i, j \leq n \text{ (by Lemma 5.1.3)}
\]
\[
\iff cf(V) \subset A_K(n, r)
\]
\[
\iff V \in M_K(n, r).
\]

We now prove that there is an equivalence of categories between the category \( M_K(n, r) \) of polynomial \( KG \)-modules and the category of \( S_K(n, r) \)-modules. This is of a capital interest, since contrary to the group algebra \( KG \) of \( G \), the Schur algebra \( S_K(n, r) \) is a finite-dimensional algebra over \( K \). Hence the study of \( S_K(n, r) \)-modules is far more pleasant than the study of polynomial \( KG \)-modules in \( M_K(n, r) \).

Theorem 5.1.5
The category of \( S_K(n, r) \)-modules and the category \( M_K(n, r) \) of polynomial \( KG \)-modules are equivalent.

Proof. Let \( V \in M_K(n, r) \) be a polynomial \( KG \)-module. Then \( V \) can be provided with the structure of an \( S_K(n, r) \)-module as follows: for any \( \xi \in S_K(n, r) \), choose \( \kappa_\xi \in KG \) such that \( e(\kappa_\xi) = \xi \) and let \( \xi \) act on \( v \in V \) via
\[
\xi \cdot v = \kappa_\xi v.
\]

By Lemma 5.1.2, there always exists such an element \( \kappa_\xi \), this for any \( \xi \) in \( S_K(n, r) \). Moreover, if \( \kappa_\xi \) is another element in \( KG \) satisfying \( e(\kappa_\xi') = \xi \), then \( \kappa_\xi - \kappa_\xi' \in \text{Ker}(e) \) and hence by Lemma 5.1.4, \( \kappa_\xi v = \kappa_\xi' v \), so that \( V \) is a well-defined \( S_K(n, r) \)-module.
Conversely, let $M$ be an $S_K(n,r)$-module. Then $M$ can be provided with the structure of a $KG$-module as follows: for any $\kappa \in KG$, let $\kappa$ act on $v \in V$ via

$$\kappa v = e(\kappa) \cdot v.$$  

By Lemma 5.1.4, $M \in M_K(n,r)$ and thus the proof is complete.

Finally, let $V \in M_K(n,r)$ be a polynomial left $KG$-module and consider the anti-automorphism $-1 : KG \to KG$ defined by sending any $g \in G$ to its inverse $g^{-1}$ and then extended linearly to the whole $KG$. Then the contravariant dual $V^{-1}$ of $V$ modulo $-1$ is a left $KG$-module, but not a polynomial one. However, it is possible to make the dual $V^*$ of $V$ into a left polynomial $KG$-module if we consider another anti-automorphism $\text{tr} : KG \to KG$, defined by sending any $g \in G$ to its transpose $g^\text{tr}$, extended linearly to $KG$. Using Theorem 5.1.5, one can show that this anti-automorphism of $KG$ corresponds to the anti-automorphism $\sigma$ of $S_K(n,r)$, defined on generators $\xi_{ij}$, $1 \leq i,j \leq n$, such that it sends $\xi_{ij}$ to $\xi_{ji}$ (see [Gre81] or [Mar08] for details on the $K$-basis of $S_K(n,r)$).

5.2 Irreducible $S_K(n,r)$-modules

In this section, we first give a description of the irreducible polynomial $KG$-modules in terms of partitions. Then we give a complete set in the case where $n = 2$, using the complete set of rational irreducible $KSL_2(K)$-modules given in Theorem 4.3.5. Since this was not main part of this project, we shall not do everything in details and thus refer the reader to [Gre81] or [Mar08] if interested.

5.2.1 The formal character (revisited)

Let $n,r \in \mathbb{N}$ be two non-negative integers. An $n$-composition of $r$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers such that $\sum_{i=1}^{n} \lambda_i = r$. We denote by $\Lambda(n,r)$ the set of all $n$-compositions of $r$. Now given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$, the multiplicative character of the diagonal subgroup $T$ of $G$ associated to $\lambda$ is the map $\chi_{\lambda} : T \to K^*$ sending $d(t) = \text{diag}(t_1, \ldots, t_n)$ to $t_1^{\lambda_1} \cdots t_n^{\lambda_n}$. Moreover, for $V \in M_K(n,r)$, we define the $\lambda$-weight space $V^\lambda$ of $V$ to be the set

$$V^\lambda = \{ v \in V : d(t)v = \chi_{\lambda}(d(t))v \text{ for every } d(t) \in T \}$$

and call it the weight space of weight $\lambda$ whenever it is non-zero. Now since $K$ is an algebraically closed field, one can show that any polynomial $KG$-module $V \in M_K(n,r)$ admits a decomposition (as a $K$-space)

$$V = \bigoplus_{\lambda \in \Lambda(n,r)} V^\lambda.$$  

Observe that the weight spaces verify similar properties to those verified by the weight spaces introduced in Section 5.2.1 for $KSL_2(K)$-modules. For example, the weight spaces $V^{(\lambda_1, \ldots, \lambda_n)}$ and $V^{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})}$ are isomorphic for any $\sigma \in \text{Perm}(n)$, which is a more general case of Lemma 4.2.3. In a similar way, Lemmas 4.2.4 and 4.2.5 can be generalized as well.
Definition 5.2.1
The formal character of a polynomial KG-module $V \in M_K(n,r)$ is the polynomial $\Phi \in \mathbb{Z}[x_1, \ldots, x_n]$ defined by

$$\Phi_V = \sum_{\lambda \in \Lambda(n,r)} \dim(V^\lambda) x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$  

Notice that $\Phi_V$ is homogeneous of degree $r$. Moreover, $\Phi_V$ is a symmetric polynomial by the generalization of Lemma 4.2.3. As an example, consider the $n$-dimensional $K$-vector space $E = E(n)K$ having $K$-basis $\{e_1, \ldots, e_n\}$ and consider the $r$th exterior power $\Lambda^rE$ of $E$, which we recall has $K$-basis $\{e_{i_1} \wedge \ldots \wedge e_{i_r} : 1 \leq i_1 < \ldots < i_r \leq n\}$.

Lemma 5.2.1
The formal character $\Phi_{\Lambda^rE}$ of $\Lambda^rE$ is the $r$th elementary symmetric function

$$e_r = \sum_{1 \leq j_1 < \ldots < j_r \leq n} x_{j_1} \cdots x_{j_r}.$$  

Proof. This follows immediately from straightforward calculations, which we leave to the reader. \qed

5.2.2 A complete set of irreducible $S_K(2,r)$-modules
An $n$-composition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ of $r$ is called an $n$-partition of $r$ if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ is defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$ (observe that $m = \lambda_1$). The set of all $n$-partitions of $r$ is written $\Lambda^+(n,r)$. Now if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n,r)$ is given, the Young diagram for $\lambda$ is the subset

$$[\lambda] = \{(i,j) \in \mathbb{N}^* \times \mathbb{N}^* : 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}$$

of $\mathbb{Z} \times \mathbb{Z}$. Usually, such a diagram is represented by $\lambda_i$ boxes in the $i$th row, the rows of the boxes lined up on the left. Notice that the conjugate partition $\lambda'$ of $\lambda$ is the partition corresponding to the Young diagram in which rows and columns were interchanged.

As an example, Let $n = 4$, $r = 9$ and $\lambda = (3,2,1,1)$. Then the conjugate partition $\lambda'$ of $\lambda$ is given as $\lambda' = (4,2,1)$ and the associated Young diagrams are

$$[\lambda] = \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square & \square \end{array}, \quad [\lambda'] = \begin{array}{ccc} \square & \square & \square \\ \square & \square \end{array}.$$

Lemma 5.2.2
For each $\lambda \in \Lambda^+(n,r)$, there exists an irreducible $S_K(n,r)$-module $F_\lambda$ whose character $\Phi_{F_\lambda}$ has leading term $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.  

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Proof. Choose $\lambda \in \Lambda^+(n, r)$ and let $\lambda'$ denote its conjugate partition. By the generalization of Lemma 4.2.7 and by Lemma 5.2.1, the $S_K(n, r)$-module $V = \Lambda^1 E \otimes \cdots \otimes \Lambda^r E$ has formal character
\[ \Phi_V = e_{\lambda_1} \cdots e_{\lambda_r} \]
and so has leading term $x_{1}^{\lambda_1} \cdots x_{n}^{\lambda_n}$. Hence by Lemma 4.2.4, there exists at least one composition factor $S$ of $V$ whose character has leading term $x_{1}^{\lambda_1} \cdots x_{n}^{\lambda_n}$. Setting $F_{\lambda} = S$ concludes the proof.

In fact, it turns out that the $S_K(n, r)$-module $F_{\lambda}$ described in Lemma 5.2.2 is the unique irreducible $S_K(n, r)$-module whose formal character has leading term $x_{1}^{\lambda_1} \cdots x_{n}^{\lambda_n}$. We shall not give the details of the proof, but the idea is as follows: first for a polynomial $KG$-module $V \in M_K(n, r)$, we give a relation between the formal character $\Phi_V$ of $V$ and the usual character $\chi_V$ of $V$, which for each $g \in G$ corresponds to the trace of the endomorphism $g : V \to V$. We then deduce that the list $\{\Phi_{\lambda} : \lambda \in \Lambda\}$ is linearly indpendant, where $\{V_{\lambda} : \lambda \in \Lambda\}$ forms a complete set of non-isomorphic irreducible polynomial $KG$-modules. Finally, using some facts on symmetric functions, we show that the formal characters $\Phi_{F_{\lambda}}(\lambda \in \Lambda^+(n, r))$ form a $\mathbb{Z}$-basis of the space $\text{Sym}(n, r)$ of homogeneous symmetric polynomials of degree $r$ in the indeterminates $x_1, \ldots, x_n$. Therefore, we get the following result.

**Theorem 5.2.3**

For every $\lambda \in \Lambda^+(n, r)$, the irreducible $S_K(n, r)$-module $F_{\lambda}$ described in Lemma 5.2.2 is the only irreducible $S_K(n, r)$-module whose character has leading term $x_{1}^{\lambda_1} \cdots x_{n}^{\lambda_n}$. Thus there is a one-to-one correspondence between the set of irreducible polynomial $KG$-modules in $M_K(n, r)$ and the set of $n$-partitions $\Lambda^+(n, r)$.

Now set $G = \text{GL}_2(K)$ and let $L(s, 0)$ denote the rational irreducible $KSL_2(K)$-module $L(s)$ considered as a $KG$-module. Then for a 2-partition $\lambda = (\lambda_1, \lambda_2)$ in $\Lambda^+(2, r)$, we set
\[ L(\lambda) = (\Lambda^2 E)^{\otimes \lambda_2} \otimes L(\lambda_1 - \lambda_2, 0). \]

**Theorem 5.2.4**

For every $\lambda \in \Lambda^+(2, r)$, the $S_K(2, r)$-module $L(\lambda)$ is irreducible. Moreover, the list $\{L(\lambda)\}_{\lambda \in \Lambda^+(2, r)}$ forms a complete set of non-isomorphic irreducible $S_K(2, r)$-modules.

Proof. As a $KSL_2(K)$-module, $\Lambda^2 E$ is trivial and hence $L(\lambda)$ is isomorphic to $L(\lambda_1 - \lambda_2)$ as a $KSL_2(K)$-module. Therefore, $L(\lambda)$ is irreducible as a $KG$-module by Theorem 4.3.5 and since $\text{SL}_2(K) \subset G$. Moreover, it is easy to check that $\Phi_{L(\lambda)}$ has leading term $x_{1}^{\lambda_1} x_{2}^{\lambda_2}$ and hence Theorem 5.2.3 allows us to conclude.

\[ \square \]
As an example, let \( p = 3 \). Then \( \Lambda^+(2, 6) = \{(6, 0), (5, 1), (4, 2), (3, 3)\} \) and so the irreducible \( S_\mathcal{K}(2, 6) \)-modules are \( L(6, 0) = (S^2 E)^F \), \( L(5, 1) = \Lambda^2 E \otimes E \otimes E^F \), \( L(4, 2) = \Lambda^2 E \otimes \Lambda^2 E \otimes \Lambda^2 E \) and \( L(3, 3) = \Lambda^2 E \otimes \Lambda^2 E \otimes \Lambda^2 E \).

### 5.3 Indecomposable \( K\text{SL}_2(K) \)-modules

Let \( n, r \in \mathbb{N} \) be two non-negative integers and set \( G = \text{GL}_n(K) \). For an \( n \)-composition \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r) \), we let \( S^\lambda E \) denote the polynomial \( KG \)-module \( S^{\lambda_1} E \otimes \cdots \otimes S^{\lambda_n} E \). Moreover, we let \( Z^\lambda \) be the \( K \)-subspace of \( A_K(n, r) \) consisting of all finite linear combinations of elements of the form \( z_1 \cdots z_n \) with \( z_i \in Z_i^\lambda \), where \( Z_i^\lambda = \langle X_{i,j_1} \cdots X_{i,j_v} : 1 \leq j_1, \ldots, j_v \leq n \rangle_K \). The reader can easily check that for every \( \lambda \in \Lambda(n, r) \), the \( K \)-vector space \( Z^\lambda \) is a polynomial \( KG \)-submodule of \( A_K(n, r) \). Thus the latter admits the grading

\[
A_K(n, r) = \bigoplus_{\lambda \in \Lambda(n, r)} Z^\lambda \tag{5.5}
\]

as a polynomial \( KG \)-module. Also let \( \nu \in \mathbb{N} \) be a non-negative integer and let \( 1 \leq k \leq n \) be arbitrary.

**Lemma 5.3.1**

*The KG-modules \( S^\nu E \) and \( Z_k^\nu \) are isomorphic.*

**Proof.** Consider the map \( \theta : E^{\otimes \nu} \to Z_k^\nu \) defined on generators by \( \theta(e_{j_1} \otimes \cdots \otimes e_{j_\nu}) = X_{k,j_1} \cdots X_{k,j_\nu} \). The reader can easily check that \( \theta \) is a surjective morphism of \( KG \)-modules. On the other hand, consider the morphism of \( KG \)-modules \( \psi : E^{\otimes \nu} \to S^\nu E \) given on generators by \( \psi(e_{j_1} \otimes \cdots \otimes e_{j_\nu}) = e_{j_1} \cdots e_{j_\nu} \). Clearly, \( \text{Ker} \psi = \text{Ker} \theta \) and so the result. \( \square \)

We are now able to give a very useful decomposition of \( A_K(n, r) \) as a polynomial \( KG \)-module, which was the main goal of this subsection.

**Theorem 5.3.2**

*The polynomial KG-module \( A_K(n, r) \) admits the following grading as a KG-module:*

\[
A_K(n, r) \cong \bigoplus_{\lambda \in \Lambda(n, r)} S^\lambda E.
\]

**Proof.** Let \( m : Z_1^{\lambda_1} \otimes \cdots \otimes Z_n^{\lambda_n} \to Z_1^{\lambda_1} \cdots Z_n^{\lambda_n} \) denote the multiplication isomorphism, which sends a generator \( y_1 \otimes \cdots \otimes y_n \) to \( y_1 \cdots y_n \). Then we have

\[
S^\lambda E = S^{\lambda_1} E \otimes \cdots \otimes S^{\lambda_n} E \\
\cong Z_1^{\lambda_1} \otimes \cdots \otimes Z_n^{\lambda_n} \quad \text{(by Lemma 5.3.1)} \\
\cong Z_1^{\lambda_1} \cdots Z_n^{\lambda_n} \quad \text{(via } m \text{)} \\
= Z^\lambda
\]

and Equation (5.5) yields the desired result. \( \square \)
Projective indecomposable modules

Let $K$ be an algebraically closed field, $A$ be a finite-dimensional $K$-algebra and let \( \{L(\lambda) : \lambda \in \Lambda \} \) form a complete set of non-isomorphic irreducible $A$-modules. We first give a description of the projective indecomposable $A$-modules in terms of the irreducible $A$-modules, following the approach of [Alp86]. Notice though that the same result can be proven in the more general context of finite length modules over Artin algebras, but this theory needs the notions of right minimality, essential epimorphisms and projective covers to be introduced. If interested, the reader can consult [ARO97, I: Artin Rings].

**Theorem 5.3.3**

For each $\lambda \in \Lambda$, there exists an indecomposable projective $A$-module $P(\lambda)$ containing a unique maximal $A$-submodule $M(\lambda)$ such that $P(\lambda)/M(\lambda) \cong L(\lambda)$. In fact, $P(\lambda)$ is uniquely determined by these properties and the set \( \{ P(\lambda) : \lambda \in \Lambda \} \) forms a complete set of non-isomorphic indecomposable projective $A$-modules.

**Proof.** See [Alp86, Theorem 3, pp. 31-33] for a proof of this statement.

**Remark 5.3.4**

For each $\lambda \in \Lambda$, the $A$-module $P(\lambda)$ in Theorem 5.3.3 is in fact the projective cover of the irreducible $A$-module $L(\lambda)$. We refer the reader to [ARO97, ï¿½1.4, pp. 12-14] if interested in the details.

Now by Theorem 1.2.3, any $A$-module $P$ is isomorphic to a direct sum of indecomposable $A$-modules in a unique way (up to order and up to isomorphism). If in addition $P$ is projective, any direct summand is projective as well and thus Theorem 5.3.3 yields

$$P \cong \bigoplus_{\lambda \in \Lambda} P(\lambda)^{(m_\lambda)},$$

(5.6)

where for $\lambda \in \Lambda$, the coefficient $m_\lambda$ denotes the number of times that $P(\lambda)$ appears in the decomposition (we call $m_\lambda$ the multiplicity of $P(\lambda)$ in $P$).

**Lemma 5.3.5**

Let $\lambda_1, \lambda_2$ be in $\Lambda$. Then the following assertion holds:

$$\text{Hom}_A(P(\lambda_1), L(\lambda_2)) \cong \begin{cases} K & : \text{if } \lambda_1 = \lambda_2, \\ 0 & : \text{otherwise.} \end{cases}$$

**Proof.** Let $\phi \in \text{Hom}_A(P(\lambda_1), L(\lambda_2))$ be a non-trivial morphism of $A$-modules. Since $L(\lambda_2)$ is irreducible, $\phi$ is surjective and hence $P(\lambda_1)/\ker(\phi) \cong L(\lambda_2)$, which implies the maximality of $\ker(\phi)$ in $P(\lambda_1)$ and hence Theorem 5.3.3 yields

$$L(\lambda_2) \cong P(\lambda_1)/\ker(\phi) = P(\lambda_1)/M(\lambda_1) \cong L(\lambda_1).$$

Therefore, since we assumed that \( \{ L(\lambda) : \lambda \in \Lambda \} \) forms a complete set of non-isomorphic irreducible $A$-modules, $\text{Hom}_A(P(\lambda_1), L(\lambda_2))$ is non-trivial if and only if $\lambda_1 = \lambda_2$, in which case Lemma 1.2.1 allows us to conclude.

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Corollary 5.3.6
The multiplicities $m_\lambda$ in (5.6) are given as $m_\lambda = \dim(\text{Hom}_A(P, L(\lambda)))$.

Proof. Let $\mu \in \Lambda$. Then an easy computation yields
\[
\text{Hom}_A(P, L(\mu)) = \bigoplus_{\lambda \in \Lambda} \text{Hom}_A(P(\lambda), L(\mu))^{(m_\lambda)}
\]
and applying Lemma 5.3.5 yields the desired result.

Now if $M$ is a projective $A$-module, one can easily show that $M^*$ is an injective $A$-module and thus the following result holds.

Theorem 5.3.7
For each $\lambda \in \Lambda$, there exists an indecomposable injective $A$-module $I(\lambda)$ containing a unique irreducible $A$-submodule $L(\lambda)^*$. In fact, $I(\lambda)$ is uniquely determined by these properties and the set $\{I(\lambda) : \lambda \in \Lambda\}$ forms a complete set of non-isomorphic indecomposable injective $A$-modules.

Proof. For $\lambda \in \Lambda$, let $I(\lambda) = P(\lambda)^*$ denote the linear dual of the indecomposable projective $A$-module $P(\lambda)$ given in Theorem 5.3.3 and then conclude using Lemma 1.3.1.

Finally, let $I$ denote an injective $A$-module. Then using elementary properties of the linear dual introduced in Section 1.3, we see that Equation (5.6) becomes
\[
I \cong \bigoplus_{\lambda \in \Lambda} I(\lambda)^{(m_\lambda)},
\]
where $m_\lambda = \dim(\text{Hom}_A(L(\lambda)^*, I))$, applying Equation (1.3) to Corollary 5.3.6.

5.3.2 Tackling the problem
Let $r \in \mathbb{N}$ be a non-negative integer. By Theorem 5.3.2, we have
\[
S_K(n, r)^* \cong \bigoplus_{\alpha \in \Lambda(n, r)} S^\alpha E
\]
as $S_K(n, r)$-modules, or equivalently, as polynomial $KG$-modules. Observe that $S_K(n, r)$ is projective as an $S_K(n, r)$-module and hence $S_K(n, r)^*$ is injective as an $S_K(n, r)$-module. Consequently, for $\alpha \in \Lambda^+(n, r)$, the $S_K(n, r)$-module $S^\alpha E$ is injective and thus Equation (5.7) yields
\[
S^\alpha E \cong \bigoplus_{\lambda \in \Lambda^+(n, r)} I(\lambda)^{(m_\lambda)},
\]
where $m_\lambda = \dim(\text{Hom}_G(L(\lambda)^*, S^\alpha E))$. 

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Lemma 5.3.8
Let $A$ be a finite-dimensional algebra over $K$, $0 \neq e = e^2 \in A$ and let $V$ be an $A$-module. Then the $K$-vector spaces $\text{Hom}_A(Ae, V)$ and $eV$ are isomorphic.

Proof. Consider the linear map $\phi : \text{Hom}_A(Ae, V) \to eV$ sending $\theta$ to $\theta(e)$. First $\phi$ is well-defined, since $\theta(e) = \theta(e^2) = e\theta(e) \in eV$. Moreover, $\phi(\theta) = 0$ if and only if $\theta(Ae) = 0$ and thus it is injective. Finally, consider $ev \in eV$. Then the map $\theta : se \mapsto sev$ is a well-defined morphism of $A$-modules and thus the proof is complete.

Let $\alpha \in \Lambda(n, r)$ be an $n$-composition for $r$ and let $V$ be an $S_K(n, r)$-module. Then there exists an idempotent $\xi_{\alpha} \in S_K(n, r)$ such that $V^\alpha \equiv \xi_{\alpha}V$ (see [Gre81] or [Mar08] for details).

Lemma 5.3.9
The $K$-spaces $\text{Hom}_{S_K(n, r)}(V^\ast, S^\alpha E)$ and $V^\alpha$ are isomorphic. In particular, the multiplicities in (5.8) are given as $m_{\lambda} = L(\lambda)^\alpha$, $\lambda \in \Lambda^+(n, r)$.

Proof. By Lemma 5.3.8 applied to $A = S_K(n, r)$ and $e = \xi_{\alpha}$, the $K$-spaces $\text{Hom}_{S_K(n, r)}(S_K(n, r)\xi_{\alpha}, V)$ and $V^\alpha$ are isomorphic. Then using equation (1.3) yields
$$\text{Hom}_{S_K(n, r)}(V^\ast, A_K(n, r)\xi_{\alpha}) \cong V^\alpha$$
and the reader can check that $A_K(n, r)\xi_{\alpha} = Z^\alpha$ and thus Lemma 5.3.1 concludes the proof of the first statement.

Now let $V \in M_K(2, r)$ be a polynomial $K\text{GL}_2(K)$-module, which affords the representation $\rho : \text{GL}_2(K) \to \text{GL}(V)$ and denote by $V'$ the rational $K\text{SL}_2(K)$-module whose associated representation is the restriction $\rho|_{K\text{SL}_2(K)}$. The proof of the following result is straightforward and so is left to the reader.

Lemma 5.3.10
Let $\lambda \in \Lambda(2, r)$ be a 2-decomposition of $r$. Then the $K$-spaces $V^\lambda$ and $(V')_{\lambda_1 - \lambda_2}$ are equal.

Now let $\alpha = (\alpha_1, \alpha_2) \in \Lambda(2, r)$ be a 2-partition of $r$. Then Equation (5.8) becomes
$$S^{\alpha_1}E \otimes S^{\alpha_2}E \cong \bigoplus_s I(r - s, s)^{(m_s)},$$
where $s$ runs through those non-negative integers satisfying $(r - s, s) \in \Lambda(2, r)$ and $m_s$ denotes the multiplicity $m_{r-s,s}$ of $I(r-s,s)$ in $S^\alpha E$. Moreover, applying Lemmas 5.3.9 and 5.3.10 respectively yields
$$m_s = \dim(L(r - s, s)^\alpha) = \dim(L(r - 2s)_{\alpha_1 - \alpha_2}).$$
We are now ready to give a way to compute the multiplicities $m_s$ which appear in the decomposition (5.9).
Theorem 5.3.11
The multiplicities $m_s$ in Equation (5.9) are given as

$$m_s = \begin{cases} 
1 & : r - 2\alpha_2 = \xi_0 + \ldots + p^m\xi_m, \text{for some } \xi_i \in \{a_i, a_i - 2, \ldots, -a_i\}, \\
0 & : \text{otherwise,}
\end{cases}$$

where $r - 2s = a_0 + a_1p + \ldots + a_mp^m$, for some $\xi_i \in \{a_i, a_i - 2, \ldots, -a_i\}$.

Proof. Let such an $s$ be fixed. Then by Equation (5.10), we have

$$m_s = \begin{cases} 
1 & : r - 2\alpha_2 \text{ is a weight of } L(r - 2s), \\
0 & : \text{otherwise.}
\end{cases}$$

Now by definition, $L(r - 2s) = S^{a_0}E \otimes (S^{a_1}E)^F \otimes \ldots \otimes (S^{a_m}E)^{F^m}$, where $r - 2s = a_0 + a_1p + \ldots + a_mp^m$ is the $p$-adic expansion of $r - 2s$ and whence the result.

At this point, we know how the tensor product of any two symmetric powers decomposes in terms of indecomposable rational $\text{SL}_2(K)$-modules but we have no idea of what they look like. The first step in trying to solve this problem consists in finding a way to compute their formal character. Let then $r \in \mathbb{N}$ be a fixed odd (resp. even) integer, $\alpha = (\alpha_1, \alpha_2) \in \Lambda(2, r)$ be a 2-partition of $r$, $0 \leq s \leq \alpha_2$ be an integer and denote by $m_{\alpha_2}(s)$ the multiplicity of $I(r - s, s)$ in $S^{a_1}E \otimes S^{a_2}E$. By Theorem 5.3.11, the matrix $M \in M\left(\left\lceil \frac{r}{2} \right\rceil, \left\lceil \frac{r}{2} \right\rceil\right)$ (resp. $M(\left\lceil \frac{r+2}{2} \right\rceil, \left\lceil \frac{r+2}{2} \right\rceil)$) whose coefficients satisfy $M_{ij} = m_{i-1}(j-1)$ is lower unipotent. We then have the following result.

Theorem 5.3.12
The formal characters of the indecomposable $S_K(n, r)$-modules can be computed using the recursive formula

$$\Phi_{I(r-\alpha_2, \alpha_2)} = \Phi_{S^{r-\alpha_2}E} \Phi_{S^{\alpha_2}E} - \sum_{s=0}^{\alpha_2-1} m_{\alpha_2}(s) \Phi_{I(r-s, s)}$$

Let us give a concrete application of Theorem 5.3.11 and Theorem 5.3.12, by considering an algebraically closed field $K$ of characteristic 2 and $r = 7$. First using Theorem 5.3.11, the reader can verify that the multiplicities are given as

$$m_0(s) = \begin{cases} 
1 & : s = 0, \\
0 & : \text{otherwise.}
\end{cases}$$

$$m_1(s) = \begin{cases} 
1 & : s = 0, 1, \\
0 & : \text{otherwise.}
\end{cases}$$

$$m_2(s) = \begin{cases} 
1 & : s = 0, 1, 2, \\
0 & : \text{otherwise.}
\end{cases}$$

$$m_3(s) = \begin{cases} 
1 & : s = 0, 2, 3, \\
0 & : \text{otherwise.}
\end{cases}$$
In other words, Theorem 5.3.11 yields the decompositions \( S^7E \cong I(7,0) \), \( S^6E \otimes E \cong I(7,0) \oplus I(6,1) \), \( S^5E \otimes S^2E \cong I(7,0) \oplus I(6,1) \oplus I(5,2) \) and finally \( S^4E \otimes S^3E \cong I(7,0) \oplus I(5,2) \oplus I(4,3) \). Now in terms of formal characters, we have the linear system

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\Phi_{(7,0)} \\
\Phi_{(6,1)} \\
\Phi_{(5,2)} \\
\Phi_{(4,3)} \\
\end{pmatrix}
= 
\begin{pmatrix}
\Phi_{S^7E} \\
\Phi_{S^6E} \Phi_{E} \\
\Phi_{S^5E} \Phi_{S^2E} \\
\Phi_{S^4E} \Phi_{S^3E} \\
\end{pmatrix},
\]

which is a special case of the result stated in Theorem 5.3.12 and the reader can check that we get \( \Phi_{I(7,0)} = \Phi_{S^7E} \), \( \Phi_{I(6,1)} = \Phi_{S^6E} \), \( \Phi_{I(5,2)} = \Phi_{S^5E} \) and \( \Phi_{I(4,3)} = \Phi_{S^4E} + \Phi_{E} \). In order to be efficient while doing these calculations, one can use the formal character argument used in the proof of the Clebsch-Gordan Formula 4.4.2. Notice that the formal characters \( \Phi_{I(\lambda)} \) are expressible in terms of positive sums of formal characters of symmetric powers. We shall see in the next section that this is always the case, for every \( \lambda \in \Lambda^+(2,r) \) and every \( r \in \mathbb{N} \).

Let us give an interesting consequence of Theorem 5.3.11 in characteristic two.

**Corollary 5.3.13**

Let \( \nu \in \mathbb{N} \) be a non-negative integer. Then we have

\( S^\nu E \otimes S^\nu E \cong I(\nu, \nu) \).

Hence the KG-module \( S^\nu E \otimes S^\nu E \) is always indecomposable if \( \text{char}(K) = 2 \).

**Proof.** In this case, \( \alpha_1 = \alpha_2 = \nu \), \( r = \alpha_1 + \alpha_2 = 2\nu \) and thus \( r - 2\alpha_2 = 0 \). On the other hand, for any \( s \neq \nu \), \( r - 2s \) is even and different from zero. We then conclude using Theorem 5.3.11.

This is where we would stop if there were no alternative to computing the characters of the \( I(\lambda) \)'s. However, using the theory of good filtrations, we can go further in our research.

### 5.4 Further results using \( \nabla \)-filtrations

Let \( G = \text{SL}_2(K) \) and let \( r \in \mathbb{N} \) be a non-negative integer. We denote by \( \nabla(r) \) the rational KG-module \( S^r E \). A \( \nabla \)-filtration for a rational KG-module \( V \) is a finite sequence \( V = V_0 \supset V_1 \supset \ldots \supset V_{k-1} \supset V_k = 0 \) such that for \( 0 \leq i \leq k - 1 \), the quotient \( V_i/V_{i+1} \) is either 0 or isomorphic to some \( \nabla(r_i) \), \( r_i \in \mathbb{N} \).

**Lemma 5.4.1**

Let \( V \) be a rational KG-module which admits a \( \nabla \)-filtration. Then any direct summand of \( V \) admits a \( \nabla \)-filtration as well. Furthermore, if \( W \) is another rational KG-module having \( \nabla \)-filtration, then their tensor product \( V \otimes W \) also admits a \( \nabla \)-filtration.
Hence since the symmetric powers admit good filtrations (they are themselves good filtrations), then Lemma 5.4.1 applied to Equation (5.8) tells us that $I(\lambda)$ admits a good filtration as well, this for every $\lambda \in \Lambda(2,r), r \in \mathbb{N}$. More precisely, the following assertion holds.

**Lemma 5.4.2**

Let $r \in \mathbb{N}$ and let $\lambda, \mu \in \Lambda(2,r)$. Then $(I(\lambda) : \nabla(\mu)) = [\nabla(\mu), L(\lambda)]$. Hence we have

$$
\Phi_{I(\lambda)} = \sum_{\mu \in \Lambda^+(2,r)} [\nabla(\mu) : L(\lambda)] \Phi_{\nabla(\mu)}.
$$

### 5.4.1 An example in characteristic two

Let $K$ be an algebraically closed field of characteristic two. First an easy computation yields

$$
\Phi_{\nabla(1+2r)} = \Phi_{\nabla(1)} \Phi_{\nabla(r)}^r, \quad (5.11)
$$

$$
\Phi_{\nabla(2r)} = \Phi_{\nabla(r)}^r + \Phi_{\nabla(r-1)}^r. \quad (5.12)
$$

Therefore, we have an alternative way to compute the characters of the indecomposables recursively. To illustrate this, let us consider $r = 7$. Then Equations (5.11) and (5.12) together with Theorem 4.3.2 yield $\Phi_{\nabla(7)} = \Phi_{L(7)}$ (indeed, $L(7)$ is a Steinberg module for $G$), $\Phi_{\nabla(5)} = \Phi_{L(5)} + \Phi_{L(1)}$, $\Phi_{\nabla(3)} = \Phi_{L(3)}$ (again a Steinberg $KG$-module) and $\Phi_{\nabla(1)} = \Phi_{L(1)}$. Hence by Lemma 5.4.2, the reader can check that $\Phi_{I(7,0)} = \Phi_{\nabla(7)}$, $\Phi_{I(6,1)} = \Phi_{\nabla(5)}$, $\Phi_{I(5,2)} = \Phi_{\nabla(3)}$ and $\Phi_{I(4,3)} = \Phi_{\nabla(5)} + \Phi_{\nabla(1)}$, as in the previous section.
Bibliography


